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On the continuity of the Walras correspondence in
large square economies without differentiability

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ON THE CONTINUITY OF THE WALRAS CORRESPONDENCE IN LARGE SQUARE ECONOMIES WITHOUT DIFFERENTIABILITY

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ABSTRACT. Exchange economies are defined by a mapping between an atomless space of agents and a space of characteristics where the commodity space is a separable Banach space. We characterize a concept of equilibrium stability for those economies without differentiability assumptions that rely on the continuity of the equilibrium correspondence.

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1. INTRODUCTION

Following a positive answer to the existence of competitive equilibrium, the questions that arise develop on characterizing the set of equilibria in order to analyze efficiency, uniqueness or regularity properties. This results, and specifically those of regularity, are closely related to the finiteness of the equilibrium set. It is the finite property that allows to define a concept of locally stable equilibria. For this purpose, it is required to analyze how the set of equilibria changes to small perturbations in exogenous parameters that characterizes agents and, therefore, economies. This relation between parameters and equilibrium sets has been captured in the literature through correspondences that associate economies with its equilibria. Then, the approach generally consists in proving conditions over this *equilibrium correspondence* in order to conclude that it defines finite sets. This is the aim of the pioneering work of Debreu (1970) assuming differentiability conditions and using the Theorem of Sard (1942).

One step further, Kannai (1970), Hildenbrand (1970) and Hildenbrand and Mertens (1972) introduce the study on the continuity of the *equilibrium correspondence* for pure exchange economies. All these studies understand parameters as exogenous characteristics that define the agents (i.e. consumption sets, tastes or endowments). In particular, it turns to be a crucial point the way in which a topology in the space of economies is defined.

In this study, we define a new concept of stability regarding the continuity property of the equilibrium correspondence. For this purpose, we use the concept of essential stability, that was introduced in the fixed point theory by Fort (1950) and, accordingly to game theory by Wen-Tsun and Jia-He (1962). In particular, the translation to economies says that an equilibrium is *essentially stable* if it is possible to approximate it by equilibria of “similar” economies, i.e. economies that are close to the original one under a particular metric in the space of economies that have to be precised.

Generally speaking, defining the space of economies by a metric space requires to parameterize the family of economies of interest with respect to the dimensions of similarity. In our case, the dimensions are consumption and portfolio sets, utility functions and endowments. It is possible to extend our analysis to other parameterizations of economies, e.g. including externalities, tax structures or information, by requiring that the metric space of economies remains complete.

Recently, the continuity of the equilibrium correspondence was stated by Dubey and Ruscitti (2015) and He et al. (2017). We extend their results taking into consideration infinite dimensional commodity spaces and by characterizing stability when the continuity property in the equilibrium

correspondence can not be obtained directly. In fact, our results answer the question posited in Dubey and Ruscitti (2015) about the possibility of getting stability results in infinite dimensional economies. In addition we remark that we have not restricted the economies to have the same space of agents as it has typically been done in the literature.

The rest of the paper is organized as follows. The next section develops the economy and some basic assumptions. In Section 4.2, we describe the space of economies and its topology. We characterize the continuity of the equilibrium correspondence in Section 4.4 and we state our main results of stability in Section 6.

2. THE MODEL

We characterize an economy by a relation between agents and characteristics. The characteristics of the agents and, also, the quantity of them may vary across economies. Therefore, the representation of an economy could be a map between the space of agents and the space of characteristics. In turn, it induces a distribution over the space of characteristics.

In the following subsections we define precisely the space of agents that is common to the space of economies we study and the space of characteristics.

2.1. SPACE OF AGENTS. The **space of economic agents** is an atomless measure space $(A, \mathcal{A}, \mu)$ which is also separable. That is, for any coalition $B \in \mathcal{A}$ such that $\mu(B) > 0$ there is a $\mathcal{A}$ -measurable coalition $B' \subset B$ such that $0 < \mu(B') < \mu(B)$ where $\mathcal{A}$ is separable with respect to a suitable distance.

2.2. SPACE OF CHARACTERISTICS. The commodity space is defined by a separable Banach space L whose positive cone has a non-empty (norm) interior. Consequently, the price space is given by L_+^* . Each consumption set X is a subset of L_+ . We consider the preference relation $(X, \succ)$ such that $\succ \subset X \times X$ is a transitive and irreflexive binary relation on X . Let $\mathcal{P}$ be the set preference relations. For each $(X, \succ) \in \mathcal{P}$ we associate the set $P := \{(x, y) \in X \times X \mid x \not\succ y\}$.¹ In addition we shall also consider the endowment vector e belonging to X . Finally, the **space of characteristics** is given by $\mathcal{P} \times L_+$ where one has that $((X, \succ), e) \in \mathcal{P} \times L_+$.

¹See Hildenbrand (2015) p. 96.

2.3. SPACE OF CHARACTERISTIC TYPES AND NOWHERE EQUIVALENCE. Fix a particular space of agents $(A, \mathcal{A}, \mu)$. Let $X : (A, \mathcal{A}, \mu) \rightarrow L_+$ the correspondence that associates a consumption set to each agents in the space. Consequently, denote $\succ(a) \subset X(a) \times X(a)$ the preference relation of agent $a \in A$ provided the space of agents. Given the relation $((X(a), \succ(a))$ and $x, y \in X(a)$, we shall say that $x \succ(a) y$ if and only if $(x, y) \in \succ(a)$. Similarly, define a Bochner integrable, X -valued and measurable function $e : (A, \mathcal{A}, \mu) \rightarrow L_+$ that specifies the endowments for the agents in the given space. Thus, we shall say that $e \in L_1(\mu, X)$ which allows us to well define the aggregate endowment by $\int_A e d\mu$.

We formally define an economy by means of a mapping between the space of agents and the space of characteristics.

DEFINITION 1. An economy is a measurable function with respect to a given sub- σ -algebra $\mathcal{G}$ of $\mathcal{A}$ countably generated,² $\mathcal{E} : (A, \mathcal{G}, \mu) \rightarrow \mathcal{P} \times L_+$ such that the projection on L_+ denoted by $e_{\mathcal{E}} : (A, \mathcal{A}, \mu) \rightarrow L_+$ satisfies $0 < \int_A e_{\mathcal{E}} d\mu < +\infty$.

The image of the map that defines the economy should have a measurable structure that we assume to be the σ -algebra of the Borelians on the space of characteristics. For the sake of a simple notation, we omit those precisions. The **characteristic type space** is $(A, \mathcal{G}, \mu)$ for a given economy $\mathcal{E}$ whose sub- σ -algebra is $\mathcal{G}$.

Thus, $\mathcal{E}(a) = ((X_{\mathcal{E}}(a), \succ_{\mathcal{E}}(a)), e_{\mathcal{E}}(a)) \in \mathcal{P} \times L_+$. When it is clear which mapping $\mathcal{E}$ is considered, we shall represent an agent $a \in A$ in a shorter way, that is, by $((X(a), \succ(a)), e(a)) \in \mathcal{P} \times L_+$. Even more, for the sake of simplicity, sometimes we shall write directly $((X, \succ), e) \in \mathcal{P} \times L_+$.

The image of the map that defines the economy should have a measurable structure that we assume to be the σ -algebra of the Borelians on the space of characteristics. For the sake of a simple notation, we omit those precisions. The **characteristic type space** is $(A, \mathcal{G}, \mu)$ for a given economy $\mathcal{E}$ whose sub- σ -algebra is $\mathcal{G}$.

At this point we characterize the relationship between an agent space $(A, \mathcal{A}, \mu)$ and its characteristic type space $(A, \mathcal{G}, \mu)$, where $\mathcal{G}$ is a sub- σ -algebra of $\mathcal{A}$, by introducing the nowhere equivalence condition of He et al. (2017) as follows.

Let $\mathcal{G}$ be a countably generated sub- σ -algebra of the σ -algebra $\mathcal{A}$. For any $F \in \mathcal{A}$, such that $\mu(F) > 0$, the restricted probability space $(F, \mathcal{A}^F, \mu^F)$ is defined by: $\mathcal{A}^F = \{F \cap F' : F' \in \mathcal{A}\}$ and μ^F is the probability measure rescaled from the restriction of μ to $\mathcal{A}^F$.

²By definition, a σ -algebra is countably generated if it “can be generated by countably many measurable subsets together with the null set” as in He et al. (2017) footnote 18.

We shall say that $\mathcal{A}$ is nowhere equivalent to $\mathcal{G}$ if for every $F \in \mathcal{A}$ with $\mu(F) > 0$, there exists a $\mathcal{A}$ -measurable subset F_0 of F such that $\mu(F_0 \Delta F_1) > 0$ for any $F_1 \in \mathcal{G}^F$, where $F_0 \Delta F_1$ is the symmetric difference $(F_0 \setminus F_1) \cup (F_1 \setminus F_0)$.

From an economic viewpoint, the above definition means that for a nontrivial collection of agents F , if $(A, \mathcal{A}, \mu)$ and $(A, \mathcal{G}, \mu)$ model the respective spaces of agents and characteristics types, then $\mathcal{A}^F$ and $\mathcal{G}^F$ are the sets or subcoalitions in F and the characteristic-generated subcoalitions of F respectively. Nowhere equivalence means that the σ -algebra $\mathcal{A}$ is strictly richer than its sub- σ -algebra $\mathcal{G}$ when they are restricted to the group of agents F .

2.4. WALRASIAN EQUILIBRIUM. A Bochner-integrable function $f : (A, \mathcal{A}, \mu) \longrightarrow L_+$ is an *allocation* for the economy $\mathcal{E}$ if $f \in L_1(\mu, X_{\mathcal{E}})$. Further, f is said to be *attainable* for $\mathcal{E}$ if $\int_A f d\mu = \int_A e_{\mathcal{E}} d\mu$.

The demand for agent a at prices $p \in L_+^*$ in the economy $\mathcal{E}$ is given by $D_{\mathcal{E}(a)}(p)$, i.e., maximal elements for $\succ_{\mathcal{E}}(a)$ in

$$B_{\mathcal{E}(a)}(p) = \{x \in X_{\mathcal{E}}(a) : p \cdot (x - e_{\mathcal{E}}(a)) \leq 0\}.$$

DEFINITION 2. An allocation for $\mathcal{E}$, $f : (A, \mathcal{A}, \mu) \rightarrow L_+$, is walrasian if there is a price vector $p \in L_+^*$ such that:

- (i) $f(a) \in D_{\mathcal{E}(a)}(p)$ for μ -almost all $a \in A$,
- (ii) $\int_A f d\mu = \int_A e_{\mathcal{E}} d\mu$.

Thus, a walrasian allocation jointly with a corresponding price, (f, p) , is called a walrasian equilibrium.

3. THE MAIN DIFFICULTIES OF CONSIDERING ECONOMIES WITH DOUBLE INFINITY OF AGENTS AND COMMODITIES

The aim of the present paper is to extend previous results on essential stability by allowing both infinitely many commodities and varying sets of (continuum) agents. In order to compare what has been done previously, we can identify two kind of stability results: those regarding continuity of walrasian equilibria and those concerning Cournot-Nash equilibria. In the first case, as far as we know, all commodity spaces are finite dimensional. Furthermore, sequence of economies typically have varying sets of finite agents. The consideration of a continuum of agents and an infinite dimensional commodity space is present in the game theory literature such as Correa and Torres-Martínez (2014)). We remark that these authors assume that the set of agents is fixed. Notice that even if we only consider economies with finitely many agents and an infinite dimensional

commodity space, the essential stability is an open question (See Dubey and Ruscitti (2015), p. 2).

In addition, we add (uncountable) varying sets of agents

In extending the analysis to the previously described contexts, we have the following drawbacks

- The proof for the existence of essential stability relies in Fort's Theorem (Fort (1950)) which requires a metric set of prices. However, when considering infinite dimensional spaces, price simplex is endowed with the weak*-topology which is not metrizable on the whole price space.
- Also due to Fort's Theorem, it is necessary the set of economies to be Baire. But it takes place if the set of characteristics is Polish for which it is sufficient if the commodity space is locally compact. However, Hausdorff topological vector spaces are locally compact if and only if they are finite dimensional (Aliprantis and Border (1999), Theorem 4.63, p. 150).

As for the first difficulty, if the price simplex is compact in the weak*-topology then it is metrizable. This removes the first problem. Regarding the second one, we consider a locally compact subset of a non-locally compact commodity space. In doing so, we follow two approaches; first by embedding the commodity space into the Hilbert cube which in turn is a compact subset of a topological vector space. Thus, the characteristic set belongs to a compact topological space (which is not a vector space). The second one considers a common consumption set which is weak-compact and metrizable. Again, we shall have that the characteristic set is contained in a compact set yet the commodity space is not locally compact.

4. FIRST APPROACH: COMPACTIFICATION OF THE COMMODITY SPACE.

4.1. BASIC ASSUMPTIONS AND COMPACTIFICATION OF THE COMMODITY SPACE. Given that L is a Banach separable space, it is possible to define a compactification (see Corollary 3.41 in Aliprantis and Border (1999) p. 91). Moreover, L can be understood as a subset of the compactification provided a suitable embedding.³ Thus, let (H, d_H) be the metrizable compactification of $(L, \|\cdot\|)$ where the former is a Polish space. Now, we have on L two topologies that are not directly comparables, namely, the original one, denoted simply as $\|\cdot\|$, and the induced on L by d_H , denoted $d_{H,L}$. Let $(H \times H, d_{H \times H})$ be the metrizable product space which is a compactification of the product space $L \times L$.⁴ We shall denote the relative topology on $L \times L$ as $d_{H \times H, L \times L}$.

Basic Assumptions (BA)

³More precisely, let $f : L \rightarrow H$ be an embedding between L and H . Then L is a topological subspace of H by identifying L with its image $f(L)$ which is a topological subspace of H . Recall that $f : L \rightarrow f(L)$ is an homeomorphism.

⁴Let us take $(f, f) : L \times L \rightarrow f(L) \times f(L) \subset H \times H$. From previous footnote, it is clear that (f, f) is an homeomorphism between $L \times L$ and $f(L) \times f(L)$.

For each economy $\mathcal{E}$ we have:

- (1) There exists a vector $u \in \text{int}L_+$ such that $\|u\| = 1$ and is $d_{H,L}$ -bounded.
- (2) There is a set $E \subset L_+$ which is d_H -closed and convex, satisfies $E - E \subset E$ and $X \subset E$
- (3) X is a $d_{H,L}$ -closed subset of L
- (4) $(X, \succ)$ is relatively open in $X \times X$ with respect to the metric induced by $d_{H \times H}$
- (5) Aggregate endowments are strictly positive while individual endowments belongs to a weakly compact subset B of L_+

Remark 1. *Assumption BA(1) says that there exists a reference commodity bundle u with such properties. This is a technical Assumption. BA(2) restricts all individual consumption vectors to E . This condition will allow us to define an appropriate topological structure in the set of preferences (Lemma 1 below). Condition BA(3) implies that each consumption set is closed for the topology d_H induced on L while Assumption BA(4) is a classical one but with respect to the relative original norm-topology. Finally, BA(4) restricts all total endowments to be strictly positive while individual ones belong to a common weakly compact set. Consequently, the initial endowment of each agent is norm-bounded since weakly compact sets are norm-bounded. (Diestel (1984), p. 17) An analogous assumption is made in Hart et al. (1974).*

Remark 2. *Let us note that because of Assumption BA(3) it follows that P is $d_{H \times H}$ -closed in $H \times H$ and thus it is closed in $L \times L$ for the induced topology.*

4.2. SPACE OF ECONOMIES. In this setup, two economies may have different agents rather than agents with the same name but different characteristics. Moreover, two economies may differ in size also. In order to define the space of economies and a concept of convergence in it, we state some results over the space of characteristics.

Let $\mathcal{F}(H \times H)$ be the set of all closed subsets of $H \times H$. We denote by τ_C the topology of closed convergence on $\mathcal{F}(H \times H)$. Since every P belongs to $\mathcal{F}(H \times H)$, we can define a mapping $g : \mathcal{P} \rightarrow \mathcal{F}(H \times H)$ by $(X, \succ) \mapsto P$. It is easily verified that g is an injection. Indeed, let $(X, \succ) \neq (X', \succ')$ in $\mathcal{P}$. Let us assume that $P = P'$. If $X = X'$, then we have that $X \times X \setminus \succ = X' \times X' \setminus \succ'$, whence $\succ = \succ'$ which contradicts $(X, \succ) \neq (X', \succ')$. If $X \neq X'$ one can assume without loss of generality that $X \setminus X' \neq \emptyset$. It follows from $X \times X \setminus \succ = X' \times X' \setminus \succ'$ that for any $y \in X \setminus X'$ that $(y, y) \in \succ$ which contradicts irreflexivity. Consequently, we must have $P \neq P'$ whenever $(X, \succ) \neq (X', \succ')$.

We now define a topology $\tau_C^{\mathcal{P}}$ on $\mathcal{P}$ by $\tau_C^{\mathcal{P}} = \{g^{-1}(U) : U \in \tau_C\}$. Thus $\tau_C^{\mathcal{P}}$ can be seen as the induced topology τ_C on $\mathcal{P}$.

LEMMA 1.

- (1) $(\mathcal{P}, \tau_C^{\mathcal{P}})$ is compact and metrizable.
- (2) A sequence of preferences $(X^n, \succ^n)_{n \in \mathbb{N}}$ converges to $(X, \succ)$ in $(\mathcal{P}, \tau_C^{\mathcal{P}})$ if and only if $Li(P_n) = P = Ls(P_n)$
- (3) The set $\{(X, \succ), x, y\} \in \mathcal{P} \times L \times L : x, y \in X \text{ and } x \not\succ y\}$ is closed for the product topology $\tau_C^{\mathcal{P}} \times d_{H \times H, L \times L}$.

Let us consider the set of all monotonic preference relations in $\mathcal{P}$ as follows $\mathcal{P}_{mo} := \{(X, \succ) \in \mathcal{P}, \text{ such that for all } x, y \text{ in } X, \text{ if } x \geq y \text{ and } x \neq y \text{ then } x \succ y\}$. We have the following lemma

LEMMA 2. The subset $\mathcal{P}_{mo}$ is a Polish space.

Remark 3. When $L = \mathbb{R}^\ell$ for $\ell > 0$, Hildenbrand (2015) uses the topology induced by the closed convergence on the space $\mathcal{F}(L \times L)$ rather than $\mathcal{F}(H \times H)$. This is so because $\mathbb{R}^\ell$ is locally compact.

Remark 4. Note that in the above Lemma, though we are considering the space H , the relevant topological results take place in L (or $L \times L$) with the relative topologies. Indeed, $X \subset L$ and $\succ \subset X \times X \subset L \times L$. $\mathcal{P}$ is also a subset of $L \times L$ and it is a closed subset of the latter for the topology $d_{H, L \times L}$.

We get the following Corollary.

COROLLARY 1. Let $(X, \succ) \in \mathcal{P}$ such that $x, y \in X$ and $x \succ y$. There exists an $\tau_C^{\mathcal{P}}$ -open neighborhood $U_{(X, \succ)}$ and a $d_{H \times H, L \times L}$ -open neighborhood $U_{(x, y)}$ such that for all $(X', \succ') \in U_{(X, \succ)}$ and for all $(x', y') \in U_{(x, y)} \cap (X' \times X')$ we have $x' \succ' y'$.

Proof. Since the set $\{(X, \succ), x, y\} \in \mathcal{P} \times L \times L : x, y \in X \text{ and } x \not\succ y\}$ is closed for the product topology $\tau_C^{\mathcal{P}} \times d_{H \times H, L \times L}$, then $\mathcal{P} \times L \times L \setminus \{(X, \succ), x, y\} \in \mathcal{P} \times L \times L : x, y \in X \text{ and } x \not\succ y\}$ is $\tau_C^{\mathcal{P}} \times d_{H \times H, L \times L}$ -open. $\square$

The above result allows us to take into account the case in which a small perturbation takes place in the sense that the new parameters belong to a $\tau_C^{\mathcal{P}} \times d_{H, L \times L}$ -open neighborhood of the previous ones. However, as it will become clear later,⁵ we need to consider a “mixed topology” for neighboring agent characteristics

Assumption CP (continuity of preferences). Let $(X, \succ) \in \mathcal{P}$ such that $x, y \in X$ and $x \succ y$. There exists an $\tau_C^{\mathcal{P}}$ -open neighborhood $U_{(X, \succ)}$ of $(X, \succ)$, a $d_{H, L}$ -open neighborhood of

⁵See the proof of Theorem 1

x , V_x , and an $\|\cdot\|$ - open neighborhood of y , W_y , such that for all $(X', \succ') \in U_{(X, \succ)}$ and for all $(x', y') \in (V_x \times W_y) \cap (X' \times X')$, it follows that $x' \succ' y'$.

Let us note that Assumption CP introduces a mixed topology on the product space $L \times L$ for neighboring the vector $(x, y) \in X \times X$. Actually, when comparing with Corollary 1, it changes the topology of the second coordinate of $L \times L$ by $\|\cdot\|$. The consideration of mixed-topologies has antecedents in the literature. For instance, it is a key assumption in general equilibrium theory in infinite dimensional settings. See Bewley (1972) or Yannelis and Zame (1986).

It is worth remarking that when L is locally compact we can dispense with Assumption CP since it is automatically satisfied by Corollary 1.

We state now two assumptions. Assumption C which concerns “small” perturbations of consumption sets and Assumption DC which is about identical distributions on the characteristics of an economy

Assumption C

Let $(X_n, \succ_n)$ be a sequence converging to $(X, \succ)$ with respect to τ_C^P such that $X_n, X : (A, \mathcal{A}, \mu) \rightarrow L_+$. For all $x \in L_1(\mu, X)$ and for all $\delta \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $x(a) + \delta u \in X_n(a)$ for all $n > n_0$ and all $a \in A$.

Assumption DC Let $((X, \succ), e, x)$ be a measurable mapping from an agent space $(A, \mathcal{A}, \mu)$ into the characteristics space $(\mathcal{P}_{mo} \times L_+ \times L_+)$ such that $x(a)$ and $e(a)$ belong to $X(a)$ for all $a \in A$. Let $((X', \succ'), e', x')$ be a measurable mapping from an agent space $(A', \mathcal{A}', \mu')$ into the characteristics space $(\mathcal{P}_{mo} \times L_+ \times L_+)$ such that $\mu'o((X', \succ'), e', x')^{-1} = \mu o((X, \succ), e, x)^{-1}$, then $x'(a)$ and $e'(a')$ belong to $X'(a')$ for all $a' \in A'$

By Assumption BA(2) we get that the sequence $(X_n \times X_n)$ and $(X \times X)$ belong to $\mathcal{F}(H \times H)$ and by Lemma 1.(2) $Li(P_n) = P = Ls(P_n)$. Then it follows from the irreflexivity of each $(X_n, \succ_n)$ that $Li(X_n) = X = Ls(X_n)$. Hence, Assumption C implies that $x + \delta u$ belongs to X^6 (Hildenbrand (2015) p. 15).

From an economic point of view, for a perturbed bundle in the consumption set, Assumption C ensures that the perturbed bundle is still feasible regarding a well chosen small neighborhood of the consumption set. Thus, a small change has a relatively small impact in consumption and can be

⁶For the sake of simplicity we omitted ω

counterbalanced by a small move regarding a particular direction. We make use of this Assumption in the proof of the Theorem 1.

Assumption DC means that for a given economy with certain distribution of characteristics if there is an agent space together with a measurable function that have the same distribution over the space of characteristics then we can consider the latter as another economy. It is worth noticing that when the consumption sets equal L_+ this assumption is automatically true. We make use of this condition in the proof of the Theorem 1.

Thus, Assumptions CP, C and DC may be seen as the cost to be paid for extending previous results on essential stability when considering infinite dimensional spaces which are not locally compact.

4.3. SIMILARITY BETWEEN ATOMLESS ECONOMIES. Let $\mathbb{E}$ be the set of all large economies whose agent spaces are atomless and separable and their σ -algebra are nowhere equivalent with respect to a countably generated sub- σ -algebra. The initial endowments of these economies also satisfy Assumption BA (4). Let $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ be the set of all Borel probability measures (distributions) on $\mathcal{P}_{mo} \times L_+$. Let $\mathcal{E} : (A, \mathcal{A}, \mu) \rightarrow \mathcal{P}_{mo} \times L_+$ and $\mathcal{E}' : (A', \mathcal{A}', \mu') \rightarrow \mathcal{P}_{mo} \times L_+$ be two elements of $\mathbb{E}$. Accordingly to He et al. (2017) we shall say that $\mathcal{E}$ and $\mathcal{E}'$ are similar if they are close in the sense of having similar distributions and total endowments.

In order to characterize our concept of stability, we require a metric space of economies. Therefore, we need a metrizable topology over the set of distributions. Recall that a space of probability measures is metrizable and separable if the set where the distributions are defined is separable (Billingsley (1968) Theorem 4 Appendix III). Thus, given that our space of characteristics is separable we know that there is a metric on the space of distributions of characteristics. We endow the space of distributions with the weak topology. The associated metric is given by the Prohorov metric ρ on $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ (see Billingsley (1968) Theorem 5 Appendix III). The dimension of similarity needed for aggregated endowments is given by the associated norm. Formally, we define the following metric on $\mathbb{E}$:

$$d_{\mathbb{E}}(\mathcal{E}, \mathcal{E}') = \rho(\mu o(\mathcal{E})^{-1}, \mu' o(\mathcal{E}')^{-1}) + \left\| \int_A e d\mu - \int_{A'} e' d\mu' \right\|.$$

In particular, let us note that accordingly to the above metric, two economies are said to be equal if and only if their distributions and mean edowments are equal.

PROPOSITION 1. *The space $(\mathbb{E}, d_{\mathbb{E}})$ is complete.*

4.4. WALRAS ALLOCATION CORRESPONDENCE AND WALRAS EQUILIBRIUM CORRESPONDENCE. The price simplex is given by $S = \{p \in L_+^* : p(u) = 1\}$. By Jameson (1970), Theorem 3.8.6, S is weak*-compact and since L is separable, the topology induced on S by the weak*-topology is metrizable by a translation invariant metric d_{L^*} on L^* (Dunford and Schwartz (1958), Theorem 1, p. 426). Furthermore, S is a norm bounded subset of L^* accordingly to Alaoglu's Theorem (Dunford and Schwartz (1958), Corollary 3, p. 424).

Let x belong to $L_1(\mu, L)$. Endowed with the norm $\|x\| = \int_A |x| d\mu$, $(L_1(\mu, L), \|\cdot\|)$, it is a normed linear space (Dunford and Schwartz (1958), Theorem 5, p. 121) which is also complete (Diestel (1977), p. 50) and locally convex (Schaefer (1971), p. 48). Furthermore, since every measure space concerning agents is assumed to be separable, $L_1(\mu, L)$ is a separable Banach space (Kolmogorov and Fomin (1975), p. 381). The topological dual of $L_1(\mu, L)$ is $L_\infty(\mu, L)$ and for the weak-topology $\sigma_{L_1} := \sigma(L_1(\mu, L), L_\infty(\mu, L))$ on $L_1(\mu, L)$ the space $(L_1(\mu, L), \sigma_{L_1})$ is also a locally convex topological vector space (Schaefer (1971) p. 52). It is known that we can construct an invariant metric on $L_1(\mu, L)$ that generates a weaker topology than σ_{L_1} . However, for every compact subset of $(L_1(\mu, L), \sigma_{L_1})$ both topologies induce equivalent topologies (Dunford and Schwartz (1958), Theorem 3, p. 434).

Let $\mathcal{E} : (A, \mathcal{A}, \mu) \rightarrow \mathcal{P}_{mo} \times L_+$ be an economy where $\mathcal{E}(a) = ((X(a), \succ(a)), e(a)) \in \mathcal{P}_{mo} \times L_+$ μ -a.e. $a \in A$. Then, the attainable set for the economy $\mathcal{E}$ is $\mathbb{A}(\mathcal{E}) := \{x \in L_1(\mu, X) : \int_A x d\mu = \int_A e d\mu\}$. We enunciate the next proposition in order to show that for every economy $\mathcal{E} \in \mathbb{E}$ the walrasian correspondence is contained in a compact metric set.

PROPOSITION 2. *If $X_{\mathcal{E}}$ is weak-compact valued, then for every $\mathcal{E} \in \mathbb{E}$ the set $\mathbb{A}(\mathcal{E})$ is a weakly compact metric subset of $L_1(\mu, L)$.*

Proof. First, by Diestel (1977) we know that $L_1(\mu, X_{\mathcal{E}})$ is weakly compact in $L_1(\mu, L)$ and by the above argument it is metrizable so that it is a compact metric subset of $L_1(\mu, L)$. Let $(x^n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{A}(\mathcal{E}) \subset L_1(\mu, X_{\mathcal{E}})$ which converges weakly to x . Since $\int_A x^n d\mu = \int_A e d\mu$ for every $n \in \mathbb{N}$ and the fact that the sequence (x^n) is norm-bounded, it follows by the Dominated Convergence Theorem that $\lim_n \int_A x^n d\mu = \int_A x d\mu = \int_A e d\mu$. Thus $x \in \mathbb{A}(\mathcal{E})$ and the proof is complete. $\square$

Let $\mathbb{A}$ be a subset of $\cup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})$. We endow $\mathbb{A}$ with the following metric: for x, x' in $\mathcal{E}$ and $\mathcal{E}'$ respectively such that both economies belong to $\mathbb{A}$, then $d_{\mathbb{A}}(x, x') = \rho(\mu o(x)^{-1}, \mu' o(x')^{-1})$

where ρ is the Prohorov metric on the space of probability measures $\mathcal{M}(L)$ on $(L, \mathcal{B}(L))$ ⁷. Since L is separable, ρ is a metrization of the topology of weak convergence on $\mathcal{M}(L)$. Thus, the notion of convergence we shall consider on $\mathbb{A}$ is that of convergence in distribution, i.e., a sequence of (measurable) attainable allocations (x^n) in $\mathbb{A}$ converges to the measurable allocation x if the sequence of distributions $\mu^n o x^n$ of x^n converges weakly to the distribution $\mu o x$ of x . The set $\mathbb{A}$ is assumed to be compact, therefore our definition of it should induce some equilibrium selection of feasible allocations. This is also the case in Correa and Torres-Martínez (2014) or Carbonell-Nicolau (2010). We point out that accordingly to $d_{\mathbb{A}}$, two feasible allocations in $\mathbb{A}$ are equal (even though they do not belong to the same economy) if and only if their distribution coincide.

DEFINITION 3. The Walras allocation correspondence $WA : \mathbb{E} \rightarrow \mathbb{A}$ assigns to every economy $\mathcal{E} \in \mathbb{E}$ its corresponding walrasian allocation set $WA(\mathcal{E}) \subset \mathbb{A}$

DEFINITION 4. The walrasian equilibrium correspondence $WE : \mathbb{E} \rightarrow S \times \mathbb{A}$ associates each economy $\mathcal{E} \in \mathbb{E}$ to its corresponding equilibria $WE(\mathcal{E}) \subset S \times \mathbb{A}$.

We posit the following assumption which uniformly constraints the allocations to each individual in an equilibrium point

Assumption UBE (uniform pointwise boundedness -of walrasian equilibrium points-)

Let D be a weakly compact subset of L_+ . Let $\mathcal{E}$ be an element of $\mathbb{E}$ and $(A, \mathcal{A}, \mu)$ its underlying agent space. Then, for every $x \in WA(\mathcal{E})$ it follows that $x \in L_1(\mu, D)$.

Assumption UBE says that every walrasian allocation will provide to each individual an amount of goods which are commonly upper-bounded. We note that the above result is true for each economy taken individually because of Proposition 2. Even more, it is true also in the case of large but finitely many economies. When $L = \mathbb{R}_+^l$ it follows automatically whenever individual endowments belong to a compact space and prices are strictly positive. Notice that this condition is clearly weaker than that of Bewley in Bewley (1991) which assumes that for all $x \in X(a) \subset l_\infty$ it follows that $\|x\|_\infty \leq c$ for $c > 2$. We make use of this Assumption in the proof of Theorem 1 below.

Related to the above definition there is that of Walras equilibrium distribution as stated in Hildenbrand (2015) p. 158. We adapt it to the context of our model. For $p \in S$ we define the set

$$E_p = \{(\tilde{\mathcal{E}}, \tilde{x}) \in (\mathcal{P} \times L_+) \times L_+ : \tilde{x} \in D_{\tilde{\mathcal{E}}}(p)\}$$

⁷Actually, $d_{\mathbb{A}}$ is the relativization to $\mathbb{A}$ of the metric $d_{\cup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})}(x, x') = \rho(\mu o(x)^{-1}, \mu' o(x')^{-1})$

DEFINITION 5. A Walras equilibrium distribution for distribution θ of agents' characteristics in $\mathcal{P} \times L_+$ is a probability measure τ on $\mathcal{P} \times L_+ \times L_+$ equipped with its Borel σ -algebra such that:

- (1) The marginal distribution $\tau_{\mathcal{P} \times L_+}$ equals μ
- (2) Mean demand equals mean supply
- (3) There exists $p \in S$ such that $\tau(E_p) = 1$

It is straightforward to verify that if (p, x) belongs to $WE(\mathcal{E})$, where $\mathcal{E} : (A, \mathcal{A}, \mu) \rightarrow \mathcal{P} \times L_+$ is the corresponding economy and agent space, then $\tau = \mu o(\mathcal{E}, x)^{-1}$ in $\mathcal{M}(\mathcal{P} \times L_+ \times L_+)$ is a Walras equilibrium distribution for $\mu o(\mathcal{E})^{-1}$. We are now ready for stating the next result

THEOREM 1. *The correspondence WA has a $(d_{\mathbb{E}}, d_{\mathbb{A}})$ -closed graph if Assumptions BA, CP, C, DC and UBE hold.*

Let us note that since $\mathbb{A}$ is $d_{\mathbb{A}}$ -compact the correspondence WA is $(d_{\mathbb{E}}, d_{\mathbb{A}})$ -upper hemi-continuous. We now state an immediate consequence of the above theorem

COROLLARY 2. *The correspondence WE is $(d_{\mathbb{E}}, \sigma^* \times d_{\mathbb{A}})$ -upper hemi-continuous*

We remark that not every economy in $\mathbb{E}$ may reach an equilibrium but, e.g., those that satisfy certain assumptions as those given in Khan and Yannelis (1991) or Noguchi (1997). This allows us to state the following

PROPOSITION 3. *Let $\widehat{\mathbb{E}}$ be any subset of the space $\mathbb{E}$ satisfying all assumptions of Khan and Yannelis (2013), then $WE(\mathcal{E}) \neq \emptyset$ for all $\mathcal{E}$ in the $d_{\mathbb{E}}$ -closure of $\widehat{\mathbb{E}}$*

Proof. Since Khan and Yannelis (2013) prove equilibria in each large economy with a separable Banach commodity space whose positive cone has a nonempty interior, we can guarantee that $WE(\mathcal{E}) \neq \emptyset$ for all $\mathcal{E}$ in $\widehat{\mathbb{E}}$. Let $\mathcal{E}'$ be an element of the $d_{\mathbb{E}}$ -closure of $\widehat{\mathbb{E}}$. Hence, $\mathcal{E}' = \lim_n \mathcal{E}^n$ such that $\mathcal{E}^n \in \widehat{\mathbb{E}}$ for all $n \in \mathbb{N}$. Consequently, there exists a sequence (p^n, x^n) which belongs to $WE(\mathcal{E}^n)$ for all n whence, in $S \times \mathbb{A}$. By taking a subsequence if necessary, it $\sigma^* \times d_{\mathbb{A}}$ -converges to (p, x) . By Corollary 2, $(p, x) \in WE(\mathcal{E})$ $\square$

5. SECOND APPROACH: WEAKLY COMPACT COMMODITY SPACES

In this section, instead of compactifying the commodity space L in order to define the set of preferences $\mathcal{P}$ we consider a convex and weakly compact subset F of the space L which is assumed to include all consumption sets and the vector u . Clearly, F is weak-closed (whence norm-closed)

and bounded. The norm on F , $\|\cdot\|_F$ is the induced from L . Notice that F is a Polish space since L is Polish. The positive cone of F , denoted F_+ , is $F_+ = L_+ \cap F$. The vector u belongs to the norm interior of F_+ since it belongs to $\text{int}L_+ \cap F$ and $\|u\|_F = 1$.

Close assumptions are made in large economies even if the commodity space is finite dimensional. Indeed, Hildenbrand (2015) pgs. 85-86 states a condition on consumption sets which in turn implies that the family of such spaces is a compact set (Theorem 1, p. 96). The works of Khan and Yannelis (2013) and Noguchi (1997) assume that each consumption set is weak-compact. Bewley (1991) who takes as commodity space the non-separable space l_∞ , assumes the existence of a common consumption set which is a weak*-compact subset of l_∞^+ . More recently, Khan and Sagara (2016) also assumes the existence of a common consumption set which is weak-compact and metrizable. On the other hand, F is locally compact and we point out that several papers on topologies on the space of preferences take as commodity spaces locally compact ones (see Back (1986), Chichilnisky (1980), Kannai (1970) Remark 1, Mas-Colell (1977) among others). In this sense, the consideration of F is consistent with that literature and we shall make use of some important results of that.

Assumption (BA')

For each economy $\mathcal{E}$ we have:

- (1) X contains 0 and it is contained in F_+ . It is a norm-closed and convex subset of L_+ .
- (2) $(X, \succ)$ is relatively open in $X \times X$.
- (3) Aggregate endowments are strictly positive.

Assumption BA' (1) implies that each X is a weak-closed subset of F_+ and then a weak-compact subset of L_+ . Because of Assumption BA' (2) each P is a closed subset of $L_+ \times L_+$ and then a closed subset of $F_+ \times F_+$. We now consider the set $\mathcal{F}(F \times F)$ of all closed subsets of $F \times F$. As in previous section, we denote by τ_C the topology of closed convergence on $\mathcal{F}(F \times F)$ and there is an injection $g : \mathcal{P} \rightarrow \mathcal{F}(F \times F)$ such that the topology $\tau_C^{\mathcal{P}}$ on $\mathcal{P}$ is given by $\tau_C^{\mathcal{P}} = \{g^{-1}(U) : U \in \tau_C\}$.

LEMMA 3.

- (1) $(\mathcal{P}, \tau_C^{\mathcal{P}})$ is compact and metrizable (and hence, a Polish space)
- (2) A sequence of preferences $(X^n, \succ^n)_{n \in \mathbb{N}}$ converges to $(X, \succ)$ in $(\mathcal{P}, \tau_C^{\mathcal{P}})$ if and only if $Li(P_n) = P = Ls(P_n)$
- (3) The set $\{(X, \succ), x, y\} \in \mathcal{P} \times F \times F : x, y \in X \text{ and } x \not\succ y\}$ is closed for the product topology $\tau_C^{\mathcal{P}} \times \|\cdot\|_F \times \|\cdot\|_F$. Furthermore, $\tau_C^{\mathcal{P}}$ is the weakest topology on $\mathcal{P}$ for which the above set is closed.

Proof. Since $(F \times F, \|\cdot\|_F \times \|\cdot\|_F)$ is a locally compact and separable metric space, the space $(\mathcal{F}(F \times F), \tau_C)$ is compact and metrizable (Hildenbrand (2015), Theorem 2 p. 19). Now, one can apply Theorem 1 p. 96 of Hildenbrand (2015) since it works for locally compact spaces F other than R^L . $\square$

LEMMA 4. *The subset $\mathcal{P}_{mo} \subset \mathcal{P}$ is a Polish space.*

The proof of the above result is a direct transcription of the corresponding Lemma in p. 98 of Hildenbrand (2015)

COROLLARY 3. *Let $(X, \succ) \in \mathcal{P}$ such that $x, y \in X$ and $x \succ y$. There exists an $\tau_C^{\mathcal{P}}$ -open neighborhood $U_{(X, \succ)}$, a $\|\cdot\|_F$ -open neighborhood V_x and a $\|\cdot\|_F$ -open neighborhood V_y , such that for all $(X', \succ') \in U_{(X, \succ)}$ and for all $(x', y') \in X' \cap V_x \times X' \cap V_y$ we have $x' \succ' y'$.*

The proof follows the guidelines of Corollary 1 of the previous section but replacing the topology $d_{H \times H, L \times L}$ by $\|\cdot\|_F \times \|\cdot\|_F$

5.1. SIMILARITY BETWEEN ATOMLESS ECONOMIES. Let $\mathbb{E}$ be the set of all large economies whose agent spaces are atomless and separable and their σ -algebra are nowhere equivalence with respect to a countably generated sub- σ -algebra. Let $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ be the set of all Borel probability distributions on $\mathcal{P}_{mo} \times L_+$. We endow it with the weak* topology⁸.

Since $\mathcal{P}_{mo} \times L_+$ is separable, $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ is separable and the weak* topology is metrizable by the Prohorov metric ρ (Billingsley (1968), Theorem 5 Appendix III). As in Section 4.1, for two economies $\mathcal{E} : (A, \mathcal{A}, \mu) \rightarrow \mathcal{P}_{mo} \times L_+$ and $\mathcal{E}' : (A', \mathcal{A}', \mu') \rightarrow \mathcal{P}_{mo} \times L_+$, we posit the metric:

$$d_{\mathbb{E}}(\mathcal{E}, \mathcal{E}') = \rho(\mu o(\mathcal{E})^{-1}, \mu' o(\mathcal{E}')^{-1}) + \left\| \int_A e d\mu - \int_{A'} e' d\mu' \right\|.$$

Thus, two economies are *equal* if they have the same distribution and mean endowment.

PROPOSITION 4. *The space $(\mathbb{E}, d_{\mathbb{E}})$ is complete.*

Proof. Since $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ is a separable and metrizable space, we can prove it as in Proposition 1 of Section 3. $\square$

⁸A sequence μ^n in $\mathcal{M}(\mathcal{P}_{mo} \times L_+)$ converges to the measure μ in the weak* topology $\sigma(\mathcal{M}(\mathcal{P}_{mo} \times L_+), C_b(\mathcal{P}_{mo} \times L_+))$ if and only if $\int_{\mathcal{P}_{mo} \times L_+} f d\mu^n \rightarrow \int_{\mathcal{P}_{mo} \times L_+} f d\mu$ for all $f \in C_b(\mathcal{P}_{mo} \times L_+)$, the Banach lattice of all bounded continuous real functions on $\mathcal{P}_{mo} \times L_+$.

As in Section 4.4, we have that price simplex S is weak-star compact and metrizable and each $\mathbb{A}(\mathcal{E})$ is a weak-compact and metrizable subset of $L_1(\mu, L)$. In turn, $\mathbb{A} \subset \cup_{\mathcal{E} \in \mathbb{E}} \mathbb{A}(\mathcal{E})$ is a metric space given the metric $d_{\mathbb{A}}$ of Section 4.4 and assumed compact.

THEOREM 2. *The correspondence WA has a $(d_{\mathbb{E}}, d_{\mathbb{A}})$ -closed graph if Assumptions BA' , C and DC hold.*

Let us note that unlike Theorem 1 we are not imposing neither Assumption CP nor UBE . Regarding the first one, it is because in the present framework we can make use of Corollary 3 directly (compare with comments on Assumption CP above). As for Assumption UBE , note that it is now implied by the fact that every consumption set belongs to a common weakly-compact subset F . Consequently the proof of Theorem 2 above parallels that of Theorem 1. We finish this section with the following result:

COROLLARY 4. *The correspondence WE is $(d_{\mathbb{E}}, \sigma^* \times d_{\mathbb{A}})$ -upper hemi-continuous*

Since both S and $\mathbb{A}$ are metric and compacts, both correspondences WA and WE are $(d_{\mathbb{E}}, \sigma^* \times d_{\mathbb{A}})$ -upper hemi-continuous.

6. STABILITY RESULTS

We arrive to the main section of this paper. We shall now study the stability of large economies with infinitely many commodities by analyzing how a Walras equilibrium for an economy $\mathcal{E}$ changes when their characteristics are perturbed. Formally

DEFINITION 6. Let $\mathbb{E}' \subseteq \mathbb{E}$ and $\mathcal{E} \in \mathbb{E}'$. A walrasian equilibrium (p, x) of $\mathcal{E}$ is an essential equilibrium of $\mathcal{E}$ relative to $\mathbb{E}'$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mathcal{E}' \in V(\mathcal{E}, \delta) \cap \mathbb{E}'$ it follows that $WE(\mathcal{E}') \cap V((p, x), \varepsilon) \neq \emptyset$

Thus, essential stability is equivalent to the lower hemi-continuity of the Walras equilibrium correspondence. The following definitions are taken from Kelley (1975)

DEFINITION 7. A subset A of a topological space is *nowhere dense* if and only if the interior of the closure of A is empty.

DEFINITION 8. A subset A of a topological space T is *meager* in T if and only if A is the countable union of nowhere dense sets.

DEFINITION 9. The complement of a meager set is called *co-meager* or *residual* in T

DEFINITION 10. (Schaefer (1971)) A subset A which is not meager is called *non-meager* in T . If every non-empty open subset is non-meager in T , then T is called a *Baire space*

THEOREM 3. (Baire - Kelley (1975)) *Let T be either a complete pseudo-metric space or a locally compact regular space. Then the intersection of a countable family of open dense subsets of T is itself dense in T*

The above theorem says that every locally compact space and every complete metrizable space is a Baire space. Further, it is straightforward to verify that the complement of a meager subset of complete metric space is dense. The next result is a variant of a classical result of Fort (1950) by Wen-Tsun and Jia-He (1962).

LEMMA 5. *Let T and M be a metric space and a Baire space respectively. Suppose further that $\Lambda : M \rightarrow T$ is a compact-valued and upper hemicontinuous correspondence with $\Lambda(m) \neq \emptyset$ for all $m \in M$. Then there exists a dense G_δ subset M' of M such that Λ is lower hemi-continuous at every point in M' .*

THEOREM 4. *Let $\bar{\mathbb{E}}$ be a $d_{\mathbb{E}}$ -closed subset of $\mathbb{E}$ such that $WE(\mathcal{E}) \neq \emptyset$ for all $\mathcal{E} \in \bar{\mathbb{E}}^9$. There exists a dense G_δ subset $\mathbb{E}'$ of $\bar{\mathbb{E}}$ such that the correspondence WE is l.h.c. at every point in $\mathbb{E}'$. That is to say, for any $\mathbb{E}'' \subseteq \mathbb{E}'$ any walrasian equilibrium of $\mathbb{E}''$ is essential*

Proof. Since $\bar{\mathbb{E}}$ is a closed subset of a complete space it is also complete (then Baire). Since $WE(\mathcal{E}) \neq \emptyset$ for all $\mathcal{E} \in \bar{\mathbb{E}}$ we get by Corollary 2 or Proposition 3 that WE is u.h.c and compact-valued. Furthermore, since $(\mathbb{A}, d_{\mathbb{A}})$ is a metric space we can apply Lemma 5 above so that there is dense subset $\mathbb{E}'$ of $\bar{\mathbb{E}}$ which is G_δ and such that any walrasian equilibrium in $\mathbb{E}'$ is essential $\square$

6.1. CHARACTERIZATION OF ESSENTIAL EQUILIBRIA. We propose the following properties:

- (S1) The collection of essential economies $\mathbb{E}' \subset \bar{\mathbb{E}}$ is a dense residual subset of $\bar{\mathbb{E}}$.
- (S2) If for $\mathcal{E} \in \bar{\mathbb{E}}$ we have that $WE(\mathcal{E})$ is singleton, then it is essential.
- (S3) There exists a minimal essential subset of $WE(\mathcal{E})$ for $\mathcal{E} \in \bar{\mathbb{E}}$ and any of such sets is connected.
- (S4) Given a essential and connected set $(\mathcal{E}) \subset WE(\mathcal{E})$, there exists an essential component of $WE(\mathcal{E})$ that contains $m(\mathcal{E})$.
- (S5) Every essential subset of $WE(\mathcal{E})$ is stable.

THEOREM 5. *Consider $(\mathbb{E}, d_{\mathbb{E}})$. Then, (S1) is satisfied as well as for any $\mathcal{E} \in \mathbb{E}$ we have that properties (S2)-(S5) hold.*

⁹By Corollary 2 or 4 and Proposition 3, we know that the subset $\bar{\mathbb{E}}$ exists

Proof. Given $WE(\mathcal{E}) \neq \emptyset$ for $\mathcal{E} \in \mathbb{E}$ jointly with the fact that Λ is compact-valued and upper hemi-continuous, we can apply Theorem 2 in Fort (1951) in order to achieve that there exists a dense residual subset $\mathbb{E}'$ of $\mathbb{E}$ where WE is lower-hemicontinuous (see also (Carbonell-Nicolau, 2010, Lemma 5)). Thus, as every economy $\mathcal{E} \in \mathbb{E}'$ is a point of lower-hemicontinuity of WE , it follows from (Yu, 1999, Th. 4.1, 1.) that $\mathcal{E}$ is essential with respect to $\mathbb{E}'$. With a similar argument, if $WE(\mathcal{E})$ is a singleton, the equilibrium correspondence is continuous at that point and essential relative to $\mathbb{E}$ (Yu, 1999, Th. 4.3), that is (S1). Properties (S2)-(S3) follows from applying (Yu et al., 2005, Th. 2.1). Property (S4) follows from (Yu et al., 2005, Theorem 4.1) since by its definition of stability (Def. 8 (iii)) it is sufficient to show that minimal essential sets are stable. $\square$

7. APPENDIX

7.1. Proof of Lemma 1.

- (1) Let $\{(X_n, \succ_n)\}_{n \geq 0}$ be a sequence in $\mathcal{P}$ such that it has a closed limit $(X, \succ)$. We shall prove that it belongs to $\mathcal{P}$. This is equivalent to $g(\mathcal{P})$ being closed in $\mathcal{F}(H \times H)$. Indeed, let us consider the sequence $\{g((X_n, \succ_n))\}_{n \geq 0} = \{P_n\}_{n \geq 0}$ in $\mathcal{F}(H \times H)$ where $P_n = \{(x, y) \in X_n \times X_n : x \not\succ y\}$. Since $(H \times H, d_{H \times H})$ is locally compact, $(\mathcal{F}(H \times H), \tau_C)$ is a compact metric space. Then, the sequence P_n converges to P if and only if $P = \text{Li } P_n = \text{Ls } P_n$ (Hildenbrand (2015), B.II. Theorem 2, p. 19). Let us define $X = \text{proj}_E P$ and $\succ = X \times X \setminus P$. We have to prove that $g((X, \succ)) = P$.

Let us note that for $x \in X$ it follows that $(x, x) \in P$. Indeed, for $x \in X$ there exists $x' \in E$ such that $(x, x') \in P$. Since $P = \text{Li } P_n = \text{Ls } P_n$, there exists a sequence (x_n, x'_n) belonging to P_n for each n and $\lim_n (x_n, x'_n) = (x, x')$ for the topology $d_{H \times H, L \times L}$ (Hildenbrand (2015), p. 15). Since $\succ_n$ is irreflexive for each n it follows that $(x_n, x'_n) \in P_n$ and then $(x, x') \in P$.

The above implies that X is the closed limit of the sequence (X_n) and it is nonempty since $0 \in X_n$ for all $n \in \mathbb{N}$. By following the arguments of Hildenbrand (2015), p. 97, we note that X is convex and $\succ$ is irreflexive and transitive.

Finally we only need to show that $g((X, \succ)) = P$, but it is absolutely obvious since $g((X, \succ)) = \{(x, y) \in \text{proj}_E \times \text{proj}_E : (x, y) \in P\} = P$

- (2) Follows from mimicking the proof of Theorem 1(b) of Hildenbrand (2015).
- (3) Let us note that accordingly to Theorem 1(c), p. 96 of Hildenbrand (2015), the set

$$\{((X, \succ), (x, y)) \in \mathcal{P} \times (H \times H) : x, y \in X \text{ and } x \not\succ y\}$$

is closed for the product topology $\tau_C^{\mathcal{P}} \times d_{H \times H}$. Take a sequence $((X^n, \succ^n), (x^n, y^n))$ in $\mathcal{P} \times (H \times H)$ converging to $((X, \succ), (x, y)) \in \mathcal{P} \times (H \times H)$. It is clear that (x^n, y^n) converges to (x, y) for $d_{H \times H}$. Moreover, since each (x^n, y^n) belongs to $X^n \times X^n$ and $(x, y) \in X \times X$, then the sequence actually converges for $d_{H \times H, L \times L}$.

7.2. Proof of Lemma 2. We follow very closely the guidelines of Hildenbrand (2015) Lemma p. 98 but in the light of our model. Indeed, we shall prove first that $\mathcal{P}_{mo}$ with the metric of the closed convergence is a G_δ -set, i.e., a countable intersection of open sets in $\mathcal{P}$. Without loss of generality we may assume that (H, d_H) is normable. Indeed, we may consider that H is the Hilbert cube whose metric can be obtained from a norm $\|\cdot\|_H$ on H . Thus, for $m \in \mathbb{N}$ let us define the set $\mathcal{P}_m = \{(X, \succ) \in \mathcal{P} : \text{there exists } x, y \in X, x \geq y, x \not\succ y \text{ and } \|x - y\|_{H,L} \geq \frac{1}{m}\}$ where $\|\cdot\|_{H,L}$ is the norm of H induced on L . Let $(X_n, \succ_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}_m$, then there exists a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ such that $x_n \not\succ_n y_n$ and $\|x_n - y_n\|_{H,L} \geq \frac{1}{m}$. Since $(x_n, y_n)_{n \in \mathbb{N}}$ belongs to $H \times H$ there exists a subsequence also denoted $(x_n, y_n)_{n \in \mathbb{N}}$ which converges in $d_{H \times H, L \times L}$ to (x, y) . Let $P_n = \{(x', y') \in (X_n, X_n) : x' \not\succ_n y'\}$ from which we deduce that $(x_n, y_n) \in P_n$ for each n . By Lemma 1 (2) $\text{Li}P_n = \text{Ls}P_n = P$ and then $X = \text{Li}(X_n) = \text{Ls}(X_n)$. Thus, X is $d_{H,L}$ -closed (Hildenbrand (2015), p. 15) and nonempty since 0 belongs to every X_n . Furthermore, X is a convex set since every X_n is a convex set. About the irreflexivity and transitivity of $\succ$ (see Hildenbrand (2015), p. 97). Thus $(X, \succ) \in \mathcal{P}$.

Since $(x_n, y_n) \in P_n$ for each n it follows that $(x, y) \in P$ since $P = \text{Li}P_n$. Thus $x \not\succ y$. Furthermore $x \geq y$ and $\|x - y\|_{H,L} \geq \frac{1}{m}$. Indeed, $\|x - y\|_H = \|(x - x_n) - (y - y_n) + (x_n - y_n)\|_H \geq \|x_n - y_n\|_H - \|(y - y_n) + (x - x_n)\|_H \geq \frac{1}{m} - \|(y - y_n) + (x - x_n)\|_H$. Since $y_n \rightarrow y$ and $x_n \rightarrow x$ it follows that $\|x - y\|_H \geq \frac{1}{m}$. Consequently, $(X, \succ) \in \mathcal{P}_m$ whence $\mathcal{P}_m$ is closed.

Let us note that $\mathcal{P}_{mo} = \bigcap_{m=1}^{\infty} (\mathcal{P} \setminus \mathcal{P}_m)$ and thus $\mathcal{P}_{mo}$ is a G_δ -set. Then, by the classical Alexandroff lemma (see Aliprantis and Border (1999), p.86), $\mathcal{P}_{mo}$ is a Polish space.

7.3. Proof of Proposition 1. Let $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ be a Cauchy sequence on $\mathbb{E}$ where, by definition, each $\mathcal{E}^n$ is a $\mathcal{G}^n$ -measurable function from $(A^n, \mathcal{A}^n, \mu^n)$ into $\mathcal{P}_{mo} \times L_+$, being $\mathcal{G}^n$ a countably generated sub- σ -algebra of $\mathcal{A}^n$ for which the latter is nowhere equivalent to $\mathcal{G}^n$ for all n . We have that $\{\mu^n o(\mathcal{E}^n)^{-1}\}_{n \in \mathbb{N}}$ and $\{\int_{A^n} e^n d\mu^n\}_{n \in \mathbb{N}}$ are also Cauchy sequences on $\mathcal{M}(\mathcal{P} \times L_+)$ and L_+ respectively. Since $(\mathcal{M}(\mathcal{P}_{mo} \times L_+), \rho)$ is complete there exists a measure $\delta \in \mathcal{M}(\mathcal{P}_{mo} \times L_+)$ such that $\lim_n \rho(\mu^n o(\mathcal{E}^n)^{-1}, \delta) = 0$ and since $(L, \|\cdot\|)$ is a Banach space there exists a vector z in L such that $\lim_n \|\int_{A^n} e^n d\mu^n - z\|_L = 0$. Since L_+ is τ -closed we have that $z \in L_+$. Thus the sequence $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ converges. It only remains to show that it does in $\mathbb{E}$, that is to say, that there exists a

$\mathcal{G}$ -measurable function $\mathcal{E}$ from $(A, \mathcal{A}, \mu)$ to $\mathcal{P}_{mo} \times L_+$ such that $(X(a), \succ(a), e(a)) \in \mathcal{P}_{mo} \times L_+$ for all $a \in A$, $\mu o \mathcal{E}^{-1} = \delta$, $\int_A e d\mu = z$, and $\mathcal{G}$ being a sub- σ -algebra of $\mathcal{A}$ that is countably generated for which $\mathcal{A}$ is nowhere equivalent.

Let us consider the following measure space $(A, \mathcal{A}, \mu)$ where $A = (\mathcal{P}_{mo} \times L_+) \times [0, 1]$, $\mathcal{A} = \mathcal{B}(\mathcal{P}_{mo} \times L_+) \otimes \mathcal{B}([0, 1])$ and $\mu = \delta \otimes \lambda$ where λ is the Lebesgue measure. Thus, $\mathcal{E} : (A, \mathcal{A}, \mu) \rightarrow \mathcal{P}_{mo} \times L_+$ is the standard representation of δ which induces the sub- σ -algebra $\mathcal{G} = \mathcal{B}(\mathcal{P}_{mo} \times L_+) \otimes \{[0, 1], \emptyset\}$ of $\mathcal{A}$.¹⁰ It can be shown that $(A, \mathcal{A}, \mu)$ is atomless, $\mathcal{A}$ is nowhere equivalent to $\mathcal{G}$ which, in turn, is countably generated since $(\mathcal{P}_{mo} \times L_+) \times [0, 1]$ is second countable. Furthermore, $\mathcal{E}$ is $\mathcal{G}$ -measurable and it is the (distributional) limit of the sequence $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$. Furthermore, since $(\mathcal{P}_{mo} \times L_+) \times [0, 1]$ is a Hausdorff space, $\mathcal{B}(\mathcal{P}_{mo} \times L_+) \times [0, 1]$ separates points and then it is separable (Dudley (1999), Theorem 5.3.1, p. 186). Since both $\mathcal{P}_{mo} \times L_+$ and $[0, 1]$ are separable, $\mathcal{B}((\mathcal{P}_{mo} \times L_+) \times [0, 1]) = \mathcal{B}(\mathcal{P}_{mo} \times L_+) \otimes \mathcal{B}([0, 1])$.

Finally, let us note that the marginal distribution $(\mu^n o (\mathcal{E}^n)^{-1})_{L_+} = \mu^n o (e^n)^{-1}$ converges weakly to the marginal distribution $(\mu o e^{-1})$, each e^n is integrable and e is $\mathcal{A}$ -measurable. By Skorokhod's Theorem, there exist a measure space $(\Omega, \mathfrak{A}, \nu)$ and measurable mappings $\hat{\mathcal{E}}_n = (\hat{X}^n, \hat{\succ}^n, \hat{e}^n)$ and $\hat{\mathcal{E}} = (\hat{X}, \hat{\succ}, \hat{e})$ from $(\Omega, \mathfrak{A}, \nu)$ into $\mathcal{P}_{mo} \times L_+$ such that (i) $\hat{\mathcal{E}}_n$ converges a.e. to $\hat{\mathcal{E}}$, (ii) $\nu o \hat{\mathcal{E}}_n^{-1} = \mu^n o \mathcal{E}_n^{-1}$, and $\nu o \hat{\mathcal{E}}^{-1} = \mu o \mathcal{E}^{-1}$. In consequence, since $\hat{\mathcal{E}}_n$ and $\hat{\mathcal{E}}$ are measurable, then $\hat{e}_n$ and $\hat{e}$ as projection on L_+ are measurable. Moreover, the convergence given by (i) implies that $\nu o \hat{e}_n^{-1}$ converges to $\nu o \hat{e}^{-1}$ and the distributions in (ii) imply that $\nu o \hat{e}_n^{-1} = \mu^n o e_n^{-1}$ and $\nu o \hat{e}^{-1} = \mu o e^{-1}$. By Assumption BA(4) $\hat{e}_n(\omega)$ belongs to B , then it is norm-bounded for each n and then

$$\lim_{n \rightarrow \infty} \int_{\Omega} \hat{e}_n(\omega) d\nu = \int_{\Omega} \hat{e}(\omega) d\nu,$$

¹⁰For details on the standard representation, see Hildenbrand (2015) p. 156.

by the Dominated Convergence Theorem (Dunford and Schwartz (1958), Th. 10 p.328). Thus, using repeatedly a change of variables and Dominated Convergence:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{A_n} e_n d\mu_n &= \lim_{n \rightarrow \infty} \int_{\mathcal{B}(L_+)} d(\mu_n o e_n^{-1}) \\
&= \lim_{n \rightarrow \infty} \int_{\mathcal{B}(L_+)} d(\nu o \widehat{e}_n^{-1}) \\
&= \lim_{n \rightarrow \infty} \int_{\Omega} \widehat{e}_n d\nu = \int_{\Omega} \widehat{e}_d \nu \\
&= \int_{\mathcal{B}(L_+)} d(\nu o \widehat{e}^{-1}) \\
&= \int_{\mathcal{B}(L_+)} d(\mu o e^{-1}) = \int_A e d\mu
\end{aligned}$$

8. PROOF OF THEOREM 1

Regarding the closed graph property let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence which converges weakly to $\mathcal{E}$ in the sense that $\lim_n \rho(\mu_n o \mathcal{E}^n, \mu o \mathcal{E}) = 0$ and $\lim_n \left\| \int_{A_n} e_n d\mu_n - \int_A e d\mu \right\| = 0$. Let $(x^n)_{n \in \mathbb{N}}$ be a sequence such that $x^n \in WA(\mathcal{E}^n)$ for all $n \in \mathbb{N}$ and which $d_{\mathbb{A}}$ -converges. It means that $\mu^n o x_n^{-1} \xrightarrow{\rho} \gamma$ for $\gamma \in \mathcal{M}(L)$. We want to prove that there exists an allocation $x \in \mathbb{A}$ such that $\mu o x^{-1} = \gamma$. We notice that a similar result is proved in Theorem 1 of He et al. (2017) but with a finite dimensional commodity space. In their proof, the authors make use of a lemma of Keisler and Sun (2009), Lemma 2.1 (iii), p. 1586, and the fact that the nowhere equivalence holds. Applied to our economy, that lemma works provided that the space $\mathcal{P}_{mo} \times L_+ \times L_+$ is Polish. Indeed, this is the case accordingly to Lemma 2 and the fact that L_+ is a closed subsets of the Polish space L . Consequently we can follow the guidelines of He et al. (2017) to obtain an allocation $x \in \mathbb{A}$ such that $\mu o x^{-1} = \gamma$. In addition we have that the sequence $\tau^n = \mu^n o (\mathcal{E}^n, x^n)^{-1}$ ($n \in \mathbb{N}$) converges to $\tau = \mu o (\mathcal{E}, x)^{-1}$ and the marginals $\tau_{\mathcal{P} \times L_+}$ and τ_{L_+} are $\mu o \mathcal{E}^{-1}$ and $\mu o x^{-1}$ respectively.

Now, we shall prove that $x \in WA(\mathcal{E})$. In doing so we shall make use of Skorokhod's Theorem (Billingsley (1968)) to the sequence $\tau^n \rightarrow \tau$. So, there is a measure space $(\Omega, \mathcal{O}, \nu)$ and measurable mappings $(\hat{\mathcal{E}}^n, \hat{x}^n)$ ($n \in \mathbb{N}$) and $(\hat{\mathcal{E}}, \hat{x})$ from $(\Omega, \mathcal{O}, \nu)$ into $\mathcal{P}_{mo} \times B \times L_+$ such that $(\hat{\mathcal{E}}^n, \hat{x}^n)_{n \in \mathbb{N}}$ converges a.e. in Ω to $(\hat{\mathcal{E}}, \hat{x})$, $\tau^n = \nu o (\hat{\mathcal{E}}^n, \hat{x}^n)^{-1}$ ($n \in \mathbb{N}$) and $\tau = \nu o (\hat{\mathcal{E}}, \hat{x})^{-1}$. Let us note that by Assumption DC, $(\hat{\mathcal{E}}^n, \hat{x}^n)$ and $(\hat{\mathcal{E}}, \hat{x})$ may be seen as economies-allocations pairs. Since $\tau^n = \nu o (\hat{\mathcal{E}}^n, \hat{x}^n)^{-1} = \mu^n o (\mathcal{E}^n, x^n)^{-1}$ and considering the fact that there exists a price sequence $(p^n)_{n \in \mathbb{N}} \in S$ such that $(p^n, x^n) \in WE(\mathcal{E}^n)$ for all n , we deduce that $(p^n, \hat{x}^n)$ is a Walras equilibrium for $\hat{\mathcal{E}}^n$ for all $n \in \mathbb{N}$ whence $\hat{x}^n \in WA(\hat{\mathcal{E}}^n)$ for all $n \in \mathbb{N}$. The argument is identical to the one in the last paragraph of this proof. Since $(\hat{\mathcal{E}}^n, \hat{x}^n)$ converges pointwise to $(\hat{\mathcal{E}}, \hat{x})$ we claim that

$\hat{x} \in WA(\hat{\mathcal{E}})$. Indeed, take into account that $\int_{\Omega} \hat{x}^n d\nu = \int_{\Omega} \hat{e}^n d\nu$ for all $n \in \mathbb{N}$. Now the sequence $\hat{x}^n$ belongs to $\mathbb{A}$ and by Assumption UBE, $\hat{x}^n(\omega)$ is norm-bounded a.e ω in Ω . Thus we can apply the Dominated Convergence Theorem to get $\int_{\Omega} \hat{x} d\nu = \lim_n \int_{\Omega} \hat{x}^n d\nu$. Since $\hat{e}^n$ converges pointwise to $\hat{e}$ and it belongs to B we use Assumption BA(3) to deduce again by the Dominated Convergence Theorem that $\int_{\Omega} \hat{e} d\nu = \lim_n \int_{\Omega} \hat{e}^n d\nu$. Hence, $\int_{\Omega} \hat{x} d\nu = \int_{\Omega} \hat{e} d\nu$. Furthermore, since the sequence $(p^n)_{n \in \mathbb{N}}$ belongs to S there is a subsequence also denoted by (p^n) which converges to $p \in S$ in the weak*-topology. Hence, since $(\hat{x}^n(\omega))$ and $(\hat{e}^n(\omega))$ norm converge to $\hat{x}(\omega)$ and $\hat{e}(\omega)$ respectively a.e $\omega \in \Omega$, $(p, \hat{x}) \mapsto p \cdot \hat{x}$ and $(p, \hat{e}) \mapsto p \cdot \hat{e}$ are jointly continuous. Hence, $p^n \cdot \hat{x}^n = p^n \cdot \hat{e}^n$ for all n implies $p \cdot \hat{x} = p \cdot \hat{e}$. Finally, we show that $\hat{x}(\omega) \in D_{\hat{\mathcal{E}}(\omega)}(p)$ a.e $\omega \in \Omega$. Suppose not, then there exists $\xi \in L_1(\nu, \hat{X})$ such that $\xi(\omega) \succ_{\omega} \hat{x}(\omega)$ and $p \cdot \xi(\omega) < p \cdot \hat{e}(\omega)$ for ω in a non-null subset of Ω . Take the real number $\gamma > 0$ such that $p \cdot \xi(\omega) + \gamma < p \cdot \hat{e}(\omega)$. Choose $0 < \delta < \gamma$ such that $\xi(\omega) + \delta u \succ_{\omega} \hat{x}(\omega)$ by Corollary 1. Because of Assumption C and since $\text{Li}(X^n(\omega), \succ_{\omega}^n) = (X(\omega), \succ_{\omega}) = \text{Ls}(X^n(\omega), \succ_{\omega}^n)$ there exists $n_0 \in \mathbb{N}$ such that $\xi(\omega) + \delta u \in X_n(\omega)$ for all $n > n_0$. By Assumption CP there exists $n_1 \in \mathbb{N}$ such that $\xi(\omega) + \delta u \succ_{\omega}^n \hat{x}^n(\omega)$ for all $n > n_1$. Because of equilibrium conditions in $\hat{\mathcal{E}}_n$ it follows that $p^n \cdot (\xi(\omega) + \delta u) > p^n \cdot \hat{e}^n(\omega)$ for all $n \in \mathbb{N}$ such that $n > n_0$ and $n > n_1$. Taking limits we get $p \cdot \xi(\omega) + \delta \geq p \cdot \hat{e}(\omega)$ which contradicts the above converse inequality.

Consequently, $\tau = \nu o(\hat{\mathcal{E}}, \hat{x})^{-1}$ is a Walras equilibrium distribution for $\nu o \hat{\mathcal{E}}^{-1}$ which is equal to $\nu o(\hat{\mathcal{E}}, \hat{x})^{-1}$ (see Hildenbrand (2015), Definition 5, p. 158). Therefore, $\nu o \hat{\mathcal{E}}^{-1} = \mu o \mathcal{E}^{-1}$ that is to say, both economies have the same distribution. Hence, $\nu o \hat{e}^{-1} = \mu o e^{-1}$ where $\hat{e}$ is the endowment of the economy $\hat{\mathcal{E}}$. By Lemma 8, p. 182, in Dunford and Schwartz (1958) we get $\int_{\Omega} \hat{e} d\nu = \int_A e d\mu$ which means that both economies $\mathcal{E}$ and $\hat{\mathcal{E}}$ have the same mean endowment.

Recall that $(\hat{x}, p) \in WE(\hat{\mathcal{E}})$ implies $\int_{\Omega} \hat{x} d\nu = \int_{\Omega} \hat{e} d\nu$. In addition, note that the marginal $\tau_{L+} = \nu o \hat{x}^{-1} = \mu o x^{-1}$ whence, again by Lemma 8, p. 182, in Dunford and Schwartz (1958), we obtain $\int_{\Omega} \hat{x} d\nu = \int_A x d\mu$ from which we get $\int_A x d\mu = \int_A e d\mu$ since mean endowments are equal.

Let us note that $\nu o(\hat{\mathcal{E}}, \hat{x})^{-1}(E_p) = 1$ which implies that $\mu o(\mathcal{E}, x)^{-1}(E_p) = 1$. Thus $\mu\{a \in A : (\mathcal{E}(a), x(a)) \in E_p\} = 1$ which implies that, a.e $a \in A$, $x(a) \in D_{\mathcal{E}(a)}(p)$. Hence (p, x) is a walrasian equilibrium for $\mathcal{E}$.

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