ASOCIACION ARGENTINA
DE ECONOMIA POLITICA

ANALES | ASOCIACION ARGENTINA DE ECONOMIA POLITICA

L Reunión Anual

Noviembre de 2015

ISSN 1852-0022
ISBN 978-987-28590-3-9

Pareto dominant strategy-proof rules with multiple commodities

Bonifacio, Agustín

Pareto dominant strategy-proof rules with multiple commodities*

Agustín G. Bonifacio†

Abstract

We study *strategy-proof* allocation rules in economies with a social endowment of perfectly divisible commodities and multidimensional single-peaked preferences. Using the property of *replacement monotonicity,* we: (i) establish sufficient conditions for a rule to be *Pareto dominant strategy-proof;* (ii) present a multidimensional version of the sequential rules introduced by Barberà, Jackson and Neme [6] and show that they also are *Pareto dominant strategy-proof;* and (iii) give a new characterization of the multidimensional uniform rule with this notion of Pareto domination. These results generalize previous work of Anno and Sasaki [4], that only applies to the two-agent case.

Resumen

Estudiamos reglas de asignación *no manipulables* en economías con una dotación social compuesta de bienes perfectamente divisibles y preferencias unimodales multidimensionales. Usando la propiedad de *monotonía en reemplazos*: (i) establecemos condiciones para que una regla sea *Pareto dominante entre reglas no manipulables;* (ii) presentamos una extensión multidimensional de las reglas secuenciales introducidas por Barberà, Jackson y Neme [6] y mostramos que estas reglas también son *Pareto dominantes entre reglas no manipulables;* y (iii) damos una nueva caracterización de la regla uniforme multidimensional que utiliza esta noción de dominación de Pareto. Estos resultados generalizan trabajo previo de Anno y Sasaki [4], que sólo considera economías de dos agentes.

Journal of Economic Literature Classification Numbers: D63, D71, D78

*I would like to thank Alejandro Neme for his guidance and patience. I also would like to thank William Thomson for his hospitality. Of course, all errors are my own. This work is partially supported by Universidad Nacional de San Luis and by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET). Part of this research was undertaken while visiting the Department of Economics, University of Rochester, under a Fulbright Scholarship.

†Instituto de Matemática Aplicada San Luis (UNSL-CONICET), Av. Italia 1556, 5700, San Luis, Argentina. E-mail: abonifacio@unsl.edu.ar

1 Introduction

The literature on *strategy-proofness* on specific economic environments, in contradistinction to abstract public goods common in social choice theory, begins with the pioneering work of Hurwicz [13]. In private goods contexts, Hurwicz considers a class of two-good two-agent economies with individual endowments and classical preferences and shows that no rule is *strategy-proof* (which means that the rule selects allocation in such a way that no agent ever benefits from lying about his preference relation), *efficient* (which means that the rule selects an allocation such that no other allocation pareto dominates it), and *individually rational* (which means that, for each economy, a rule assigns to each agent a bundle that he finds at least as desirable as his endowment). Dasgupta, Hammond and Maskin [10] and Zhou [23] generalize this result weakening *individual rationality* and then Serizawa [21] extends it to economies with several goods and several agents.[1]

Similar results were obtained in economies with a social endowment and more restricted preference domains, in which it is shown that *strategy-proof* and *efficient* rules are *dictatorial* (the requirement that there be no agent who receives everything for each economy). For example, such statement holds in two-agent economies with homothetic or linear preferences (Schummer [20]), with constant elasticity of substitution preferences (Ju [14]) or with quasi-linear preferences (Goswami et al [12]). Recent results in economies with several agents are presented by Momi [15]. Cho and Thomson [9] find that, in economies with more than two agents and linear preferences, no rule is *strategy-proof, efficient* and *equally treating* (which means that the rule should assign indifferent bundles to agents with the same preferences). These results show that one of the cornerstones of social choice theory, the celebrated Gibbard-Satterhwaite Theorem (Gibbard [11], Satterhwaite [19]), still holds in private good economic environments, where alternative sets are endowed with a variety of mathematical structures and, therefore, admissible preference relations are restrained considerably.[2] The Gibbard-Satterthwaite Theorem essentially says that, with public goods, every *strategy-proof* rule is *dictatorial* on the universal domain, i. e., when all preferences are admissible.[3]

We study economies with a social endowment of perfectly divisible commodities and multidimensional single-peaked preferences defined over those commodities. These preferences fulfill the following requirement: for each of the goods in the economy, keeping fixed the level of consumption of the other goods, an increase in an agent's consumption raises his welfare up to some critical level; beyond that level, the opposite holds. The importance of the single-peaked preferences domain comes from the facts that: (i) under

[1]A related result is established by Cho [8]. He introduces, for $\delta \in [0,1]$, the properties of δ-*strategy-proofness* and δ-*efficiency*. These parametrizations are equivalent to the original properties when $\delta = 1$ and weaken monotonically, eventually to a vacuous requirement when $\delta = 0$. His principal result establishes that, in two-agent economies defined on a domain containing linear preferences, for each $\delta \in (0,1]$, no rule is: (i) *strategy-proof*, δ-*efficient* and *individually rational*, and (ii) δ-*strategy-proof, efficient* and *individually rational.*

[2]An important exception is the domain of preferences representable by Leontief-type utility functions, in which there are *strategy-proof* and *efficient* rules that are not *dictatorial* (Nicoló [18]).

[3]Notice, however, that when we restrict ourselves to the domain of single-peaked preferences, the family of median voter rules are *strategy-proof, efficient* and *anonymous* (which means that the rule is invariant with respect to the names of the agents) (See Moulin [17]).

some regularity conditions,[4] this domain generalizes the one of classical preferences, and (ii) several interpretations of the model can be made in different applications.

One of those applications is the study of general (dis)equilibrium models, as in Bénassy [7]. Consider an exchange economy with classical preferences where goods are assigned through the price mechanism (Walrasian rule), but assume that prices are not in equilibrium, either because of an exogenous shock or because they are kept fixed in order to fulfill some social goal. When the distribution is made, not every agent can maximize his preferences within his budget set, so the implementation of a rationing scheme is needed. Notice, however, that if the preferences of the agents are strictly convex, their restriction to the boundary of their budget set will be multidimensional single-peaked.

Another application consists of imagining a group of partially altruist agents who have to divide among them a bundle of goods. Each agents cares about what he gets but also cares about what everyone else gets. A natural specification of the preferences of an agent in this context can be as follows: (i) when the agents consumes little he pays more attention to himself, meaning by this that an increase in his own consumption improves his welfare, and (ii) as his consumption increases, however, the agent gives more priority to the consumption of the rest. At some point, his interest for the other agents can dominate his private consumption and its increase may worsen his welfare. This preferences, therefore, are also multidimensional single-peaked.

The starting point in the literature on private good economies and single-peaked preferences is the work of Sprumont [22], in which the allocation of one perfectly divisible good is studied. In contrast with the impossibility results obtained in economies with classical preferences, non trivial *strategy-proof* and *efficient* rules exist in this case. The class of *strategy-proof* and *efficient* rules is very large, as shown by Barberà, Jackson and Neme [6]. They study rules that allow for an asymmetric treatment of the agents, generalizing the uniform rule, and characterize the family of sequential rules as the only *strategy-proof* and *efficient* rules that also fulfill the property of *replacement monotonicity.* This last property can be seen as a strong notion of solidarity, and requires that, when the preferences of an agent change (in a non-disruptive way) the welfare of the rest of the agents is affected in the same direction.

The study of allocation rules in economies with several goods and multidimensional single-peaked preferences begins with Amorós [3]. He shows that, as in classical economies, the combination of *strategy-proofness* and *efficiency* es troublesome: every rule satisfying both properties is also *dictatorial,* even in two-agent economies. Therefore, if we want to study *strategy-proof* rules, we need a weaker notion of *efficiency.*

Anno and Sasaki [4] study which rules are "the most efficient ones" in the class of *strategy-proof* rules, by means of a notion of domination between rules. Their main result states that, in two-agent economies, the multidimensional uniform rule is the only *Pareto dominat strategy-proof rule* that also fulfills *equal treatment.*

The goal of this paper is threefold: (i) to establish sufficient conditions for a rule to be *Pareto dominant strategy-proof;* (ii) to present a multidimensional version of the sequential rules introduced by Barberà, Jackson and Neme [6] and show that they also are *Pareto*

[4] Basically, continuity and strict convexity.

dominant strategy-proof; and (iii) to give a new characterization of the multidimensional uniform rule with this notion of Pareto domination. These results generalize previous work of Anno and Sasaki [4], that only applies to the two-agent case.

The paper is organized as follows. In Section 2 we introduce the model, the properties we will analyze and a general impossibility theorem. The *Pareto dominant strategy-proof rules* are introduced and some related results presented in Section 3. The multidimensional sequential rules and the multidimensional uniform rule are analyzed in Sections 4 and 5, respectively. Finally, some comments are gathered in Section 6.

2 Model, properties and preliminary results

Let $\boldsymbol{L} \equiv \{1, \ldots, l\}$ be the set of goods. For each $\ell \in L$, there is an amount $\boldsymbol{\Omega_\ell} \in \mathbb{R}_+$ to allocate. Let $\boldsymbol{\Omega} \equiv (\Omega_1, \ldots, \Omega_l) \in \mathbb{R}_+^L$ be the social endowment. The consumption set is then $\boldsymbol{X} \equiv \prod_L [0, \Omega_\ell]$. Let $\boldsymbol{N} \equiv \{1, \ldots, n\}$ be the set of agents. Each agent $\boldsymbol{i} \in N$ has a preference relation which is complete, transitive, continuous and strictly convex $\boldsymbol{R_i}$ defined on X. Denote the strict and indifference relations associated to R_i by $\boldsymbol{P_i}$ and $\boldsymbol{I_i}$, respectively. In general, we denote a domain of preferences by $\boldsymbol{\mathcal{R}}$. A profile of preferences is a list $\boldsymbol{R} \equiv (R_1, \ldots, R_n) \in \mathcal{R}^N$. Given a profile $R \in \mathcal{R}^N$ and an agent $i \in N$, the notation $\boldsymbol{R_{-i}}$ refers to the list of preferences of all the agents except agent i. We will use a similar notation replacing i with a set of agents $S \subset N$. An **economy** is a pair $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+^L$. A **(feasible) allocation** is a list $\boldsymbol{x} \equiv (x_1, \ldots, x_n) \in X^N$ such that $\sum_N x_i = \Omega$. Let $\boldsymbol{Z}$ be the set of feasible allocations. An **(allocation) rule**, denoted by $\boldsymbol{\varphi}$, is a function from $\mathcal{R}^N$ to Z. Throughout this paper we will kept fixed the social endowment $\Omega \in \mathbb{R}_+^L$. Therefore, an economy will simply be represented by a profile of preferences $R \in \mathcal{R}^N$.

Definition 1 *A preference R_i is multidimensional single-peaked if there is a $p(R_i) \in X$ such that, for each pair $x_i, x_i' \in X$ with $x_i \neq x_i'$, whenever for each $\ell \in L$, either $x_{i\ell}' \leq x_{i\ell} \leq p_\ell(R_i)$ or $p_\ell(R_i) \leq x_{i\ell} \leq x_{i\ell}'$, we have $x_i \; P_i \; x_i'$.*

From now on $\boldsymbol{\mathcal{R}}$ stands for the domain of multidimensional single-peaked preferences.

Remark 1 The domain of classical preferences, i.e., preferences satisfying monotonicity, continuity and strict convexity, is a subdomain of $\boldsymbol{\mathcal{R}}$.

Figure 1 shows a typical preference from $\mathcal{R}$, while Figure 2 show two preferences which are not multidimensional single-peaked.

Next we present the properties of *strategy-proofness, efficiency* and *equal treatment.* The first one limits the strategic behavior of the agents by requiring that none of them can manipulate the rule declaring false preferences:

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R_i' \in \mathcal{R}$, we have $\varphi_i(R) \; R_i \; \varphi_i(R_i', R_{-i})$.

The next property requires that, for each economy, a rule select an allocation such that no other allocation Pareto dominates it.

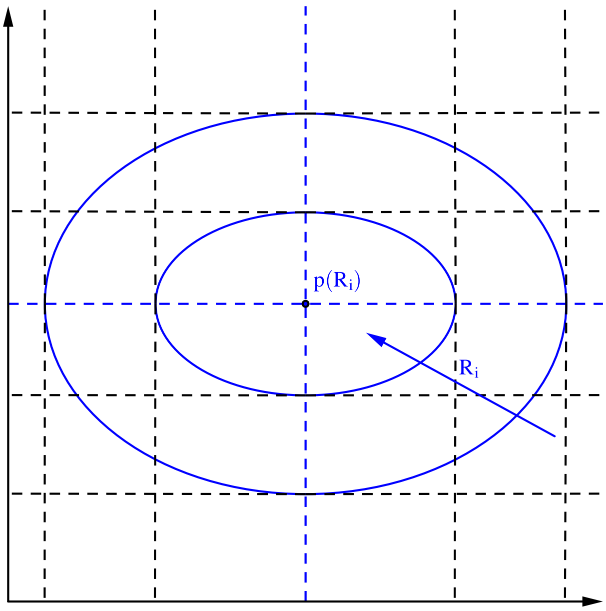

Figure 1: **Preferences belonging to the domain $\mathcal{R}$.**

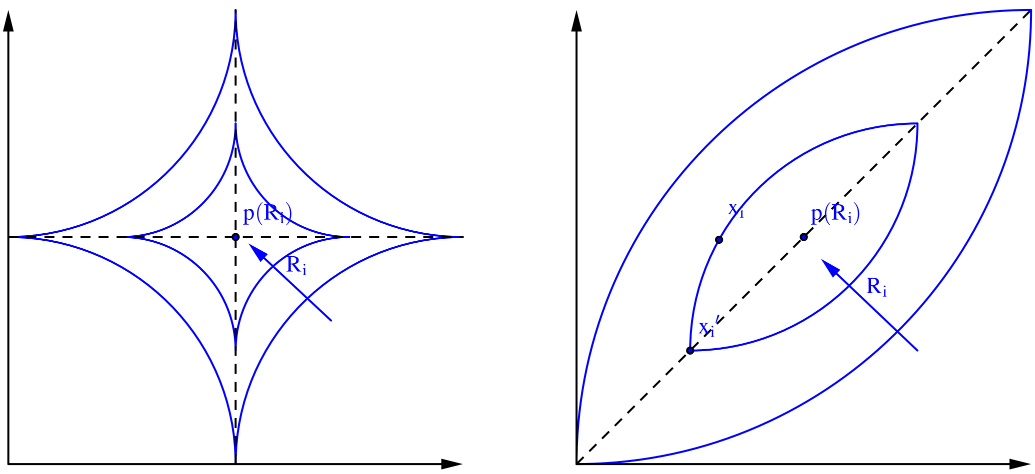

Figure 2: **Preferences that do not belong to the domain $\mathcal{R}$.** Preferences on the left are not convex. The ones on the right are not separable.

Efficiency: For each $R \in \mathcal{R}$, there is no $x' \in Z$ such that, for each $i \in N$, $x'_i \; R_i \; \varphi_i(R)$, and for some $j \in N$, $x'_j \; P_j \; \varphi_j(R)$.

The following property establishes that two agents with the same preferences must receive indifferent allocations.

Equal treatment: For each $R \in \mathcal{R}^N$ and each $\{i, j\} \subset N$, if $R_i = R_j$ then $\varphi_i(R) \; I_i \; \varphi_j(R)$.

In the domain of classical, homothetic and smooth preferences, Serizawa [21] shows that these three properties are incompatible. Let $\mathcal{R}_{cl}$ be the domain of classical, homothetic and smooth preferences.[5]

Theorem 1 (Serizawa, 2002) *No rule defined on $\mathcal{R}_{cl}^N$ is strategy-proof, efficient and equally-treating.*

The next theorem hows that the imposibility result extends tho the context of multi-dimensional single-peaked preferences.

Theorem 2 *No rule defined on $\mathcal{R}^N$ is strategy-proof, efficient and equally-treating.*

Proof. This is a straightforward consequence of the fact that $\mathcal{R}_{cl} \subset \mathcal{R}$. Let φ be a *strategy-proof, efficient* and *equally-treating* rule defined on $\mathcal{R}$. Sea $\hat{\varphi}$ the restriction of φ to $\mathcal{R}_{cl}$. Then $\hat{\varphi}$ is *strategy-proof, efficient* and *equally-treating* on $\mathcal{R}_{cl}$. This contradicts Theorem 1. $\square$

As *strategy-proofness* and *efficiency* are incompatible together with a fairness property such as *equal treatment,* and since we are interested in studying *strategy-proof* rules, we have to weaken *efficiency.* The following property does that (and is equivalent to *efficiency* in one-good economies):

Same-sidedness: For each $R \in \mathcal{R}^N$ and each $\ell \in L$, (i) if $\sum_N p_\ell(R_i) \geq \Omega_\ell$, then for each $i \in N$, $\varphi_{i\ell}(R) \leq p_\ell(R_i)$; and (ii) if $\sum_N p_\ell(R_i) \leq \Omega_\ell$, then for each $i \in N$, $\varphi_{i\ell}(R) \geq p_\ell(R_i)$.

Of course, every *efficient* rule is *same-sided,* but the converse is false.[6] As we previously commented, this property was used by Amorós [3] to characterize the multidimensional version of the uniform rule as the only *strategy-proof* and *equally treating* rule that also fulfills this weak efficiency notion.

An even weaker property says that when the sum of the peaks of each good is equal to the social endowment of each good, then the rule must assign, for each agent and each good, that peak amount.

[5] A preference relation R_i is **homothetic** if $x_i R_i y_i$ implies $\lambda x_i R_i \lambda y_i$ for each $\lambda \in \mathbb{R}_+$, and is **smooth** if, for each $x_i \in X \setminus \partial X$, there is a unique vector in the simplex $\{q \in \mathbb{R}_+^L : ||q|| = 1\}$ which supports the upper contour set $\{y_i \in X : y_i R_i x_i\}$.

[6] Define a rule in the following way: if the sum of the peaks is greater than zero, the rules assigns proportionally with respect to the peaks; otherwise it assigns the egalitarian allocation. This rule is *same-sided.* Consider a two-good, two-agent economy such that the peak amount of both agents equals the social endowment and there es a feasible allocation that dominates the egalitarian allocation. In this economy, the rule selects the egalitarian allocation. However, such allocation is not *efficient.*

Unanimity: For each $R \in \mathcal{R}^N$, if for each $\ell \in L$, $\sum_N p_\ell(R_i) = \Omega_\ell$, then for each $i \in N$ and each $\ell \in L$, $\varphi_{i\ell}(R) = p_\ell(R_i)$.

Clearly, every *same-sided* rule is *unanimous,* but the converse is false.[7] This property can be consider as a minimal requirement of efficiency.

Next we present two properties that will be fundamental for the results of this paper. The first one establishes that, for each good, if the change in the preferences of one agent does not decrease (increase) the amount assigned to that agent, then the amounts assigned to the rest of the agents cannot increase (decrease).

Replacement monotonicity: For each $R \in \mathcal{R}^N$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $\ell \in L$, (i) if $\varphi_{i\ell}(R) \leq \varphi_{i\ell}(R'_i, R_{-i})$, then for each $j \in N \setminus \{i\}$, $\varphi_{j\ell}(R) \geq \varphi_{j\ell}(R'_i, R_{-i})$; and (ii) if $\varphi_{i\ell}(R) \geq \varphi_{i\ell}(R'_i, R_{-i})$, then for each $j \in N \setminus \{i\}$, $\varphi_{j\ell}(R) \leq \varphi_{j\ell}(R'_i, R_{-i})$.

The second property, introduced by Satterthwaite and Sonnenschein [**?**], has played an important role in the development of the axiomatic study of resource allocation. It establishes that, if the change in the preferences of one agent does not change the amount assigned to that agent, then nobody else's assignment should change.

Non-bossiness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$, then $\varphi(R) = \varphi(R'_i, R_{-i})$.

Remark 2 Every *replacement monotonic* rule is *non-bossy.* When there are only two agents, any rule satisfies trivially *replacement monotonicity* (and therefore *non-bossiness*).

Several different interpretations has been given to *non-bossiness.* From a strategic perspective, this property strengthens, in some models, the property of *strategy-proofness* to *group strategy-proofness*[8]. From a normative perspective, the property has the advantage of dismiss rules with certain "arbitrary" behavior.

With respect to *replacement monotonicity,* when there is only one good in the economy, the property can be interpreted as a solidarity principle: the welfare of the agents must be affected in the same direction before (non-disruptive) changes in the preferences of one of them. Even though this idea is compelling to invoke the property, the most important justification in models with several goods is a technical one: to give some structure to classes of rules that on themselves are very difficult to describe, such the class of *strategy-proof* rules.

3 Pareto dominant strategy-proof rules

Here we present a way to weaken the property of efficiency within the class of *strategy-proof* rules. The idea is to select those rule which are undominated in the sense of Pareto. Next we present the formal definitions. Let $\mathbf{\Phi}$ denote the class of all the rules with domain $\mathcal{R}^N$.

[7]Think of a rule which assigns the peaks when feasible but is constant otherwise.

[8]*Strategy-proofness* and *non-bossiness* imply *group strategy-proofness,* for example, in one-good economies with single-peaked preferences. However, when there are several goods, this no longer holds. The multidimensional uniform rule, which we present in Section AGREGAR is not *group strateg-proof* (see Example 1). For a thorough account of the use of *non-bossiness* in the literature, see Thomson [**?**].

Definition 2 *Let φ and ψ be two rules in Φ. Then* $\boldsymbol{\varphi}$ **dom** $\boldsymbol{\psi}$ *if and only if for each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi_i(R)\ R_i\ \psi_i(R)$.*

Remark 3 The relation *dom* is a preorder, i.e., it is reflexive and transitive, but not necessarily complete nor antisymmetric.[9]

The property of *efficiency* can be characterized in terms of the relation *dom* as follows:

Lemma 1 *A rule $\varphi \in \Phi$ is efficient if and only if, for each $\psi \in \Phi$ such that ψ dom φ, we have φ dom ψ.*

Proof. ($\Longrightarrow$) Let $\varphi \in \Phi$ be an *efficient* rule and let $\psi \in \Phi$ be such that ψ *dom* φ. Assume it is not the case that φ *dom* ψ. Then there are $R \in \mathcal{R}^N$ and $i \in N$ such that $\psi_i(R)\ P_i\ \varphi_i(R)$. Since $\psi_j(R)\ R_j\ \varphi_j(R)$ for each $j \in N \backslash \{i\}$, then φ is not *efficient*. Absurd. ($\Longleftarrow$) Assume that for each $\psi \in \Phi$ such that ψ *dom* φ, we have φ *dom* ψ, and that φ is not *efficient*. Then there are $R' \in \mathcal{R}^N$ and $x \in Z$ such that, for each $j \in N$, we have $x_j\ R'_j\ \varphi_j(R')$ and there is $i \in N$ such that $x_i\ P'_i\ \varphi_i(R')$. Define the following rule:

$$\psi'(R) \equiv \begin{cases} x & \text{si} \quad R = R' \\ \varphi(R) & \text{si} \quad R \neq R', \end{cases}$$

Then ψ' *dom* φ. However, φ does not dominate ψ'. □

This means that a rule φ is *efficient* whenever it is a maximal element of Φ preordered by *dom*. Therefore, a way to weaken this property consists of constraining the set over which the rule must be maximal. We will focus our analysis on the class of *strategy-proof* rules that also fulfill the following informational simplicity property:

Own peak-onlyness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $p(R'_i) = p(R_i)$, then $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$.

Let $\boldsymbol{\Phi}^*$ denote the class of *strategy-proof* and *own peak-only* rules. We now present the notion of efficiency among *strategy-proof* rules due to Anno and Sasaki [4].[10]

Definition 3 *A rule $\varphi \in \Phi^*$ is Pareto-dominant strategy-proof if, for each $\psi \in \Phi^*$ such that ψ dom φ, we have φ dom ψ.*

Clearly, this property is implied by *efficiency* when the rule is *strategy-proof.* A more demanding version of the property requires the rule to be a maximal element on Φ^* with respect to the preorder *dom* and that no other rule be welfare-equivalent to it.[11]

Definition 4 *A rule $\varphi \in \Phi^*$ is strongly Pareto-dominant strategy-proof if, for each $\psi \in \Phi^*$ such that ψ dom φ, we have $\varphi = \psi$.*

[9] A binary relation $\prec$ defined on a set X is **reflexive** if, for each $x \in X$, we have $x \prec x$; it is **transitive** if, for each $\{x, y, z\} \subset X$, $x \prec y$ and $y \prec z$ imply $x \prec z$; it is **complete** if, for each $\{x, y\} \subset X$, we have $x \prec y$ or $y \prec x$; and it is **antisymmetric** if, for each $\{x, y\} \subset X$, $x \prec y$ and $y \prec x$ imply $x = y$.

[10] Anno and Sasaki [4] named this property *second-best efficiency.* We prefer not to use this terminology.

[11] The rules φ and ψ are **welfare-equivalent** whenever φ *dom* ψ and ψ *dom* φ.

Theorem 3 in Anno and Sasaki [4] establishes that, in two-agent economies (i.e., when $|N| = 2$), every *strategy-proof* and *same-sided* rule is *strongly Pareto-dominant strategy-proof.* The next theorem generalizes the aforementioned result to economies with an arbitrary amount of agents.

Theorem 3 *Let φ be a strategy-proof, unanimous and replacement monotonic rule. Then φ is a strongly Pareto-dominant strategy-proof rule.*

Proof. See the Appendix. □

Remark 4 Theorem 3 in Anno and Sasaki [4] is implied by Theorem 3 since: (i) by Remark 2, when $|N| = 2$, *replacement monotonicity* is trivially satisfied by any rule; and (ii) *unanimity* and *replacement monotonicity* imply *same-sidedness* (see Lemma 4 in the Appendix).

4 Multidimensional sequential rules

In this section we study a multidimensional generalization of the sequential rules presented in Barberá, Jackson y Neme [6]. The main result in that paper says that a rule is sequential if and only if it is *strategy-proof, same-sided* and *replacement monotonic* (recall that in one-good economies *same-sidedness* is equivalent to *efficiency*). We will build multidimensional rules based on this one-dimensional rules. Given $R \in \mathcal{R}^N$, let $\boldsymbol{\pi_\ell(R)}$ be the **projection of $\boldsymbol{R}$ onto the $\boldsymbol{\ell}$-th good** (which is, of course, a one-dimensional single-peaked preference).

Multidimensional sequential rule, $\boldsymbol{\phi}$: For each $R \in \mathcal{R}$ and each $i \in N$,

$$\phi_i(R) \equiv (\phi_{i\ell}(\pi_\ell(R)))_{\ell \in L},$$

where, for each $\ell \in L$, $\phi_\ell = (\phi_{i\ell})_{i \in N}$ is a one-dimensional *strategy-proof, same-sided* and *replacement monotonic* rule.

Theorem 4 *Every multidimensional sequential rule is strategy-proof, same-sided and replacement monotonic.*

Proof. See the Appendix. □

The following result is immediate consequence of Theorems 3 and 4.

Corollary 1 *Every multidimensional sequential rule is strongly Pareto-dominant strategy-proof.*

The next property establishes that each agent must find his assignment at least as desirable as the egalitarian allocation.

Egalitarian lower bound: For each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi_i(R)\ R_i\ \frac{\Omega}{n}$.

In the following we show how to construct multidimensional sequential rules that meet the *egalitarian lower bound.* Given $\ell \in L$, a **one-dimensional sequential rule $\boldsymbol{\phi_\ell}$**

(Barberà, Jackson and Neme [6]) is defined through a sequential adjustment function[12] $g_\ell : Z_\ell \times \pi_\ell(\mathcal{R}^N) \to Z_\ell \times \pi_\ell(\mathcal{R}^N)$ (that fulfills some requirements) and an initial reference vector $q^*_\ell \in Z_\ell$, by means of the following relation:

$$\phi_{i\ell}(\pi_\ell(R)) \equiv q_{i\ell}, \text{ where } q_{i\ell} \text{ comes from } (q_{i\ell}, \pi_\ell(R_i)) = g^n_\ell(q^*_\ell, \pi_\ell(R_i)),$$

and g^n_ℓ denotes the n-th iteration of g_ℓ. It is easy to see that, given a sequential adjustment function $g = (g_\ell)_{\ell \in L}$, if we choose as reference vector the egalitarian allocation (this is, if we take for each $\ell \in L$, $q^*_\ell = \frac{\Omega_\ell}{n}$), the associated sequential rule $\phi = (\phi_\ell)_{\ell \in L}$ meets the *egalitarian lower bound.*

5 The multidimensional uniform rule

A special case of multidimensional sequential rule is the multidimensional uniform rule. This rule is defined through a "non-discriminatory" sequential adjustment function (in each stage it offers the same to all of the agents) and has as initial reference vector the egalitarian allocation.

The multidimensional uniform rule is defined applying the uniform rule commodity by commodity.

Multidimensional uniform rule, $\boldsymbol{U}$: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $\ell \in L$,

$$U_{i\ell}(R) \equiv \begin{cases} \min\{p_\ell(R_i), \lambda_\ell(R)\} & \text{if } \sum_N p_\ell(R_j) \geq \Omega_\ell \\ \max\{p_\ell(R_i), \lambda_\ell(R)\} & \text{if } \sum_N p_\ell(R_j) \leq \Omega_\ell, \end{cases}$$

where $\lambda_\ell(R) \geq 0$ and solves $\sum_N U_{j\ell}(R) = \Omega_\ell$.

The rule determines some bounds, that at the same time determine a kind of common "budget set" in which each agent has to maximize his preferences. The bounds that define such set (that geometrically is a $|L|$-dimensional box) are chosen in such manner that feasibility is assured (see Figure 3). This rule is *strategy-proof* (see Amorós [3]), *same-sided* (and therefore *unanimous*), *own peak-only* and *replacement monotonic.* However, it is not *efficient.* This is shown in the following example.

Example 1 Let be $N \equiv \{1,2\}$, $L \equiv \{1,2\}$ and $\Omega \equiv (1,1)$. Let $R \in \mathcal{R}^{\{1,2\}}$ be such that $p(R_1) \equiv (1,1) \equiv p(R_2)$, $(\frac{1}{3}, \frac{2}{3})\ P_1\ (\frac{1}{2}, \frac{1}{2})$ and $(\frac{2}{3}, \frac{1}{3})\ P_2\ (\frac{1}{2}, \frac{1}{2})$. Then $U_{i\ell}(R) = \frac{1}{2}$ for each $i \in N$ and each $\ell \in L$. Let $\tilde{R} \in \mathcal{R}^{\{1,2\}}$ be such that $p(\tilde{R}_1) \equiv (\frac{1}{3}, 1)$ and $p(\tilde{R}_2) \equiv (1, \frac{1}{3})$. It follows that $U_1(\tilde{R}) = (U_{11}(\tilde{R}), U_{12}(\tilde{R})) = (\frac{1}{3}, \frac{2}{3})$ and $U_2(\tilde{R}) = (U_{21}(\tilde{R}), U_{22}(\tilde{R})) = (\frac{2}{3}, \frac{1}{3})$. This implies that, for each $i \in N$, $U_i(\tilde{R})\ P_i\ U_i(R)$. Therefore, the multidimensional uniform rule is not *efficient.*

Example 1 is illustrated in Figure 4. In this Edgeworth box preferences are specified so that: (i) make the uniform rule to recommend the egalitarian allocation, and (ii) there are feasible allocations that Pareto dominate the egalitarian allocation.

[12] Here $\pi_\ell(\mathcal{R}^N) \equiv \{\pi_\ell(R) = (\pi_\ell(R_i))_{i \in N} : R \in \mathcal{R}^N\}$ and $Z_\ell \equiv \{x_\ell = (x_{i\ell})_{i \in N} \in [0, \Omega_\ell]^N : \sum_N x_{i\ell} = \Omega_\ell\}$.

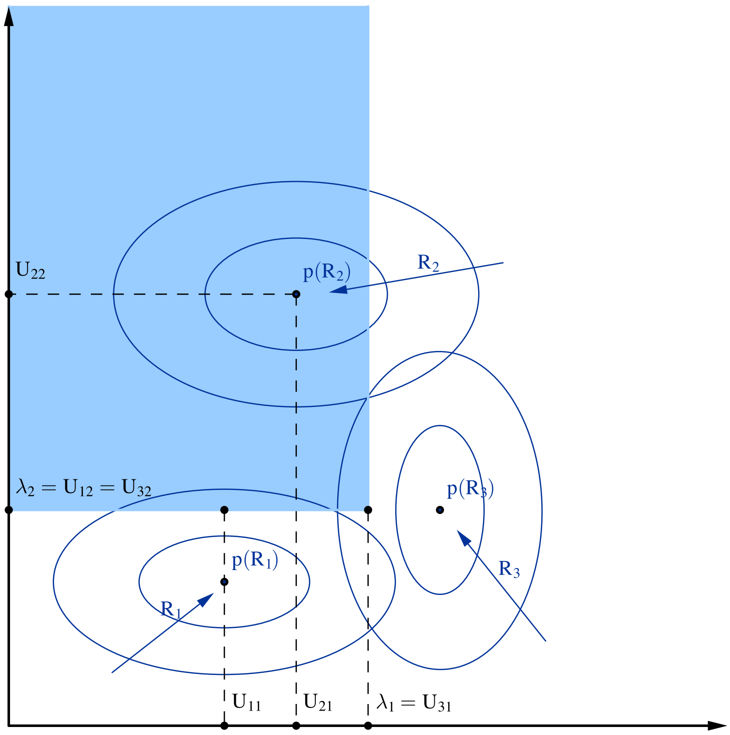

Figure 3: **The multidimensional uniform rule**. Here, $|N| \equiv 3$ and $|L| \equiv 2$, there is not enough of the first good and there is too much of the second good. Then, an upper bound $\lambda_1 \in \mathbb{R}_+$ and a lower bound $\lambda_2 \in \mathbb{R}_+$ are chosen and each agent maximizes his preferences in the rectangle $[0, \lambda_1] \times [\lambda_2, \Omega_2]$. The bounds are specified in such way that the sum of the peak amounts equals the social endowment of each good.

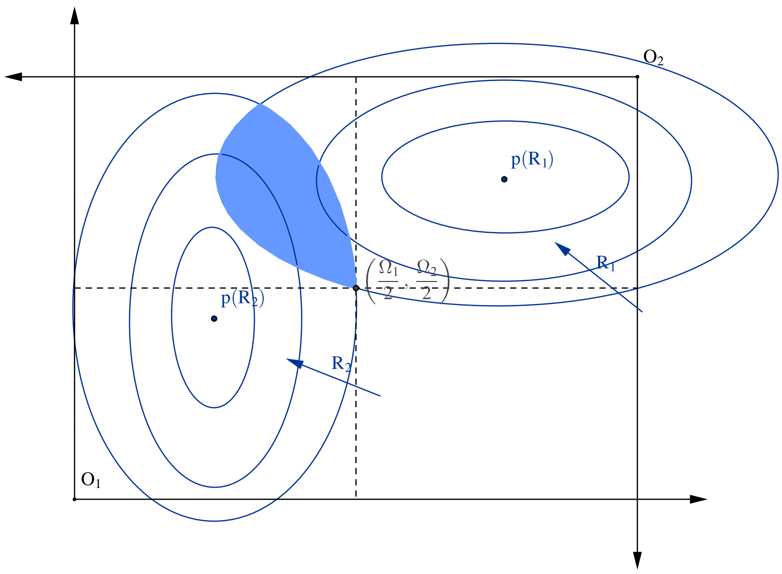

Figure 4: **The multidimensional uniform rule is not efficient**. Here, both peaks are greater than equal division for each good, so the uniform rule assigns equal division. However, there are a lot of allocations that Pareto dominate it (shaded area).

The first result concerning the multidimensional uniform rule is consequence of the rule being *strategy-proof, unanimous* and *replacement monotonic*, together with Theorem 3.

Corollary 2 *The multidimensional uniform rule is strongly Pareto dominant strategy-proof.*

Next we present a new characterization of the multidimensional uniform rule.

Theorem 5 *The multidimensional uniform rule is the only* Pareto dominant strategy-proof *rule that satisfies unanimity, equal treatment and replacement monotonicity.*

Proof. See the Appendix. □

Remark 5 the proportional rule[13] satisfies *unanimity, equal treatment* and *replacement monotonicity,* but is not *strategy-proof* and therefore is not *Pareto dominant strategy-proof.* The serial rule[14] is *(strongly) Pareto dominant strategy-proof, replacement monotonic* and *unanimous,* but does not satisfy *equal treatment.* We have not been able to

[13]**proportional rule, φ^{pro}:** For each $R \in \mathcal{R}^N$, each $\ell \in L$ and each $i \in N$,

$$\varphi^{pro}_{i\ell}(R) \equiv \begin{cases} \frac{p_\ell(R_i)}{\sum_N p_\ell(R_j)}\Omega_\ell & \text{if } \sum_N p_\ell(R_j) > 0, \\ \frac{\Omega_\ell}{n} & \text{otherwise.} \end{cases}$$

[14]**Serial rule, φ^s:** For each $R \in \mathcal{R}^N$ and each $\ell \in L$, if $i \in N \setminus \{n\}$, $\varphi^s_{i\ell}(R) \equiv \min\left\{p_\ell(R_i), \Omega_\ell - \sum_{j<i} \varphi^s_{j\ell}(R)\right\}$, and $\varphi^s_{n\ell} \equiv \Omega_\ell - \sum_{j<n} \varphi^s_{j\ell}(R)$.

find a rule that satisfies all the properties listed in Theorem 5 except *unanimity.* Also, it is an open question whether *replacement monotonicity* can be eliminated from the characterization, or at least be weakened to *non-bossiness.* Anyway, getting rid of *non-bossiness* is a non trivial issue (see Remark 2 in Morimoto, Serizawa and Ching [16] in reference to this point).

6 Final Comments

Before finishing some final remarks are in order. The property of *replacement monotonicity* has been fundamental to extending results from two-agent economies to economies with an arbitrary amount of agent (specially, to Lemma 7). However, this is a very strong property and its economic interpretation when it is not accompanied with *efficiency* is unclear. Nevertheless, we think that its use, at least in a provisional way, is justified in order to get some understanding of the structure of *strategy-proof* rules.

With respect to the multidimensional uniform rule, Anno and Sasaki [4] present three results in two-agent economies: (i) the rule is *strongly Pareto dominant strategy-proof*; (ii) the rule can be characterized through the properties of *Pareto dominant strategy-proofness, equal treatment* and *own peak-onliness;* and (iii) the rule can be characterized through *Pareto dominant strategy-proofness,* the *egalitarian lower bound* and *own peak-onliness.* Characterization (iii) is a corollary of characterization (ii) since, in two-agent economies, the *egalitarian lower bound* implies *equal treatment.*

Our more important contribution consists of answering in the affirmative the question raised by Anno y Sasaki [4] of whether their result (i) still holds in economies with an arbitrary amount of agents. Their characterization result (ii) is difficult to compare with ours since *own peak-onlyness* is part of our definition of *Pareto dominant strategy-proof* rules. They show independence of their properties. We are not able to do that.

As we already mentioned, we can construct multidimensional sequential rules meeting the *egalitarian lower bound* taking as initial reference vector the egalitarian allocation. The multidimensional uniform rule is just one of these rules, but there are plenty more (in fact, for each admissible sequential adjustment function, there is a sequential rule that meets the *egalitarian lower bound*). This implies that characterization (iii) of Anno and Sasaki [4] no longer holds with more than two agents.

A Appendix

Let be $\varphi \in \Phi$ and $i \in N$. For each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ define the **option set of agent i under φ** by $\boldsymbol{O_i^{\varphi}(R_{-i})} \equiv \{x_i \in X \mid \text{existe } R_i \in \mathcal{R} \text{ tal que } \varphi_i(R_i, R_{-i}) = x_i\}$. For each $i \in N$, each $R_i \in \mathcal{R}$, and each $Y \subseteq X$ define the **choice set of agent i on Y with respect to R_i** by $\boldsymbol{C_i(R_i, Y)} \equiv \{x_i \in Y \mid \text{para cada } y_i \in Y, x_i \; R_i \; y_i\}$.

Lemma 2 *Let φ be a rule. Then,*
(i) φ is strategy-proof *if and only if for each $R \in \mathcal{R}$ and each $i \in N$, $\varphi_i(R) \in C_i(R_i, O_i^{\varphi}(R_{-i}))$.*
(ii) If $\varphi \in \Phi^*$, then for each $i \in N$, each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, and each $\ell \in L$, there are $a_\ell, b_\ell \in [0, \Omega_\ell]$ such that $O_i^{\varphi}(R_{-i}) = \prod_L [a_\ell, b_\ell]$.

(iii) If $\varphi \in \Phi^*$, then for each $R \in \mathcal{R}$ and each $i \in N$, $C_i(R_i, O_i^{\varphi}(R_{-i})) = \{\varphi_i(R)\}$.

Proof. Part (i) es straightforward and Part (ii) is Lemma 8 in Anno and Sasaki [4]. To see Part (iii), notice that Part (ii) and single-peakedness of preferences imply $|C_i(R_i, O_i^{\varphi}(R_{-i}))| = 1$, so the result follows from Part (i). $\square$

Since option sets of *strategy-proof* and *own peak-only* rules are singletons, we often abuse notation and write, for each $\varphi \in \Phi^*_{NM}$, each $R \in \mathcal{R}$ and each $i \in N$, $C_i(R_i, O_i^{\varphi}(R_{-i})) = \varphi_i(R)$.

Domination between rules can easily be translated to an inclusion of option sets.

Lemma 3 *Let φ and ψ be two rules in Φ^*_{NM}. Then φ dom ψ if and only if for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $O_i^{\psi}(R_{-i}) \subseteq O_i^{\varphi}(R_{-i})$.*

Proof. ($\Longrightarrow$) Let $i \in N$ and $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Take $x_i \in O_i^{\psi}(R_{-i})$ and $R_i \in \mathcal{R}$such that $p(R_i) = x_i$. By Lemma 2 (iii), $\psi_i(R) = x_i$. As φ *dom* ψ, $\varphi_i(R)\ R_i\ \psi_i(R) = p(R_i)$, and therefore $\varphi_i(R) = x_i$ and $x_i \in O_i^{\varphi}(R_{-i})$.
($\Longleftarrow$) Let $i \in N$ and $R \in \mathcal{R}^N$. By Lemma 2 (i), $\varphi_i(R) \in C_i(R_i, O_i^{\varphi}(R_{-i}))$ and $\psi_i(R) \in C_i(R_i, O_i^{\psi}(R_{-i}))$. As $O_i^{\psi}(R_{-i}) \subseteq O_i^{\varphi}(R_{-i})$, $\varphi_i(R)\ R_i\ \psi_i(R)$. $\square$

The proof of Theorem 3 makes use of several lemmas. The first one is a result due to Morimoto, Serizawa and Ching [16], and states that for *strategy-proof* and *non-bossy* rules, *unanimity* is equivalent to *same-sidedness.*

Lemma 4 (Morimoto, Serizawa y Ching, 2013) *Every strategy-proof, unanimous, and non-bossy rule is same-sided.*

Proof. See Lemma 1 in Morimoto et al [16]. $\square$

The next result, due to Amorós [3], states that given three allocations $x_i^*, x_i', x_i'' \in X$, if any of the coordinates of x_i' is not in between the coordinates of x_i^* and x_i'', then x_i^* can be considered as the peak of a preference relation in which x_i'' is preferred to x_i'.

Lemma 5 (Amorós, 2002) *Let $i \in N$ and $x_i^*, x_i', x_i'' \in X$. If it is not true that, for each $\ell \in L$, $x_{i\ell}^* \leq x_{i\ell}' \leq x_{i\ell}''$ or $x_{i\ell}^* \geq x_{i\ell}' \geq x_{i\ell}''$, then there is $R_i \in \mathcal{R}$ such that $p(R_i) = x_i^*$ and $x_i''\ P_i\ x_i'$.*

Proof. See Lemma 1 in Amorós [3]. $\square$

Lemmata 6 and 7 are used to prove that every *strategy-proof, unanimous* and *replacement monotonic* rule is *own peak-only* (Lemma 8).

Lemma 6 *Let φ be a strategy-proof and non-bossy rule. Then, for each $R \in \mathcal{R}^N$, each $S \subseteq N$, each $j \in S$ and each $R_j^* \in \mathcal{R}$ such that $p(R_j^*) = \varphi_j(R)$, we have $\varphi_j(R_S^*, R_{-S}) = \varphi_j(R)$.*

Proof. Let $R \in \mathcal{R}^N$, $S \subseteq N$, $j \in S$ and $R_j^* \in \mathcal{R}$ be such that $p(R_j^*) = \varphi_j(R)$. Since φ is *strategy-proof,* $\varphi_j(R_j^*, R_{-j}) = \varphi_j(R)$ (otherwise agent j gets his peak in economy (R_j^*, R_{-j}) declaring R_j). By *non-bossiness,* $\varphi(R_j^*, R_{-j}) = \varphi(R)$. Let $k \in S \setminus \{j\}$ and $R_k^* \in \mathcal{R}$ be such that $p(R_k^*) = \varphi_k(R)$. Then, by *strategy-proofness*, $\varphi_k(R_{j,k}^*, R_{-j,k}) = \varphi_k(R_j^*, R_{-j})$ and, by *non-bossiness*, $\varphi(R_{j,k}^*, R_{-j,k}) = \varphi(R_j^*, R_{-j}) = \varphi(R)$. Continuing in the same fashion the result follows. $\square$

Given $i \in N$, $R_i, \tilde{R}_i \in \mathcal{R}$ and $x \in X$, define $\boldsymbol{L(R_i, \tilde{R}_i, x_i)} \equiv \{\ell \in L :$ either (i) $p_\ell(R_i) < x_{i\ell}$ and $p_\ell(\tilde{R}_i) \leq x_{i\ell}$, or (ii) $p_\ell(R_i) > x_{i\ell}$ and $p_\ell(\tilde{R}_i) \geq x_{i\ell}$, or (iii) $p_\ell(R_i) = p_\ell(\tilde{R}_i) = x_{i\ell}\}$.

Lemma 7 *Let φ be a strategy-proof, unanimous and replacement monotonic rule. Then, for each $R \in \mathcal{R}^N$, each $i \in N$, each $\tilde{R}_i \in \mathcal{R}$, each $\ell \in L(R_i, \tilde{R}_i, \varphi_i(R))$, each $j \in N \setminus \{i\}$ and each $R_j^* \in \mathcal{R}$ such that $p(R_j^*) = \varphi_j(R)$, we have $\varphi_{i\ell}(\tilde{R}_i, R_j^*, R_{-i,j}) = \varphi_{i\ell}(R)$.*

Proof. Let φ be a rule that satisfies the properties listed in the lemma. By Lemma 4, φ is *same-sided*, and by Remark 2, φ is *non-bossy*. Let $R \in \mathcal{R}^N$, $i \in N$ and consider the profile $R_{-i}^* \in \mathcal{R}^{N\setminus\{i\}}$ such that $p(R_j^*) = \varphi_j(R)$ for each $j \in N \setminus \{i\}$. We will prove the lemma in several steps.

Step 1: For each $\boldsymbol{i \in N}$ and each $\boldsymbol{R_i' \in \mathcal{R}}$ such that $\boldsymbol{p(R_i') = p(R_i)}$, we have $\boldsymbol{\varphi(R_i', R_{-i}^*) = \varphi(R)}$.

By Lemma 6 and *non-bossiness*, it is sufficient to see that $\varphi_i(R_i', R_{-i}^*) = \varphi_i(R_i, R_{-i}^*)$. Let $\ell \in L$. If $\varphi_{i\ell}(R_i', R_{-i}^*) \leq p_\ell(R_i')$, since φ is *same-sided* and by Lemma 6, we have $\varphi_{j\ell}(R_i', R_{-i}^*) \leq p(R_j^*) = \varphi_{j\ell}(R_i, R_{-i}^*)$ for each $j \in N \setminus \{i\}$. Therefore, $\varphi_{i\ell}(R_i', R_{-i}^*) = \Omega_\ell - \sum_{N\setminus\{i\}} \varphi_{j\ell}(R_i', R_{-i}^*) \geq \Omega_\ell - \sum_{N\setminus\{i\}} \varphi_{j\ell}(R_i, R_{-i}^*) = \varphi_{i\ell}(R_i, R_{-i}^*)$. In consequence, as $p_\ell(R_i') = p_\ell(R_i)$,

$$\varphi_{i\ell}(R_i, R_{-i}^*) \leq \varphi_{i\ell}(R_i', R_{-i}^*) \leq p_\ell(R_i). \tag{1}$$

Analogously, we can show that if $\varphi_{i\ell}(R_i', R_{-i}^*) \geq p_\ell(R_i')$, then

$$\varphi_{i\ell}(R_i, R_{-i}^*) \geq \varphi_{i\ell}(R_i', R_{-i}^*) \geq p_\ell(R_i). \tag{2}$$

Since both (1) and (2) are true for each $\ell \in L$, if $\varphi_i(R_i', R_{-i}^*) \neq \varphi_i(R_i, R_{-i}^*)$ then $\varphi_i(R_i', R_{-i}^*)\ P_i\ \varphi_i(R_i, R_{-i}^*)$, violating the *strategy-proofness* of φ.

Step 2: For each $\boldsymbol{i \in N}$, each $\boldsymbol{\tilde{R}_i \in \mathcal{R}}$ and each $\boldsymbol{\ell \in L(R_i, \tilde{R}_i, \varphi_i(R))}$, we have $\boldsymbol{\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) = \varphi_{i\ell}(R)}$.

Notice that, by Lemma 6, it is sufficient to see that $\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) = \varphi_{i\ell}(R_i, R_{-i}^*)$. Assume this is not true. Let us analyze the case in which $\ell \in L$ is such that $p_\ell(\tilde{R}_i) + \sum_{N\setminus\{i\}} p_\ell(R_j^*) \leq \Omega_\ell$, since an analogous reasoning applies to the symmetric case. We have that $p_\ell(\tilde{R}_i) \leq \Omega_\ell - \sum_{N\setminus\{i\}} p_\ell(R_j^*) = \varphi_{i\ell}(R_i, R_{-i}^*)$, since, for each $j \in N \setminus \{i\}$, $p_\ell(R_j^*) = \varphi_{j\ell}(R_i, R_{-i}^*)$. Therefore, being $\varphi_{i\ell}(R_i, R_{-i}^*) = \varphi_{i\ell}(R)$ and $\ell \in L(R_i, \tilde{R}_i, \varphi_i(R))$, we have $p_\ell(R_i) < \varphi_{i\ell}(R_i, R_{-i}^*)$. By *same-sidedness*, for each $j \in N\setminus\{i\}$, $\varphi_{j\ell}(R_i, R_{-i}^*) \geq p_\ell(R_j^*)$. Then, $\sum_{N\setminus\{i\}} \varphi_{j\ell}(\tilde{R}_i, R_{-i}^*) \geq \sum_{N\setminus\{i\}} \varphi_{j\ell}(R_i, R_{-i}^*)$ and, by feasibility, $\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) \leq \varphi_{j\ell}(R_i, R_{-i}^*)$. in consequence, we have: (i) $p_\ell(R_i) < \varphi_{i\ell}(R_i, R_{-i}^*)$ and (ii) $\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) < \varphi_{j\ell}(R_i, R_{-i}^*)$. By Lemma 5, there is $R_i' \in \mathcal{R}$ con $p(R_i') = p(R_i)$ such that $\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) P_i' \varphi_{j\ell}(R_i, R_{-i}^*)$. By Step 1, $\varphi_{j\ell}(R_i', R_{-i}^*) = \varphi_{j\ell}(R_i, R_{-i}^*)$. Thus, $\varphi_{i\ell}(\tilde{R}_i, R_{-i}^*) P_i' \varphi_{i\ell}(R_i', R_{-i}^*)$, contradicting the *strategy-proofness* of φ.

Step 3: For each $\boldsymbol{i \in N}$, each $\boldsymbol{\tilde{R}_i \in \mathcal{R}}$, each $\boldsymbol{\ell \in L(R_i, \tilde{R}_i, \varphi_i(R))}$, each $\boldsymbol{j \in N \setminus \{i\}}$ and each $\boldsymbol{R_j^* \in \mathcal{R}}$ such that $\boldsymbol{p(R_j^*) = \varphi_j(R)}$, we have $\boldsymbol{\varphi_{i\ell}(\tilde{R}_i, R_j^*, R_{-i,j}) = \varphi_{i\ell}(R)}$.

By Step 2, we only need to show that $\varphi_{i\ell}(\tilde{R}_i, R_j^*, R_{-i,j}) = \varphi_{i\ell}(\tilde{R}_i, R_{-i}^*)$. First, we show that

$\varphi_{i\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) = \varphi_{i\ell}(\tilde{R}_i, R^*_{-i})$. By *non-bossiness,* it is sufficient to see that $\varphi_{j\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) = \varphi_{j\ell}(\tilde{R}_i, R^*_{-i})$. Assume, without loss of generality, that $\varphi_{j\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) < \varphi_{j\ell}(\tilde{R}_i, R^*_{-i})$. This means, by *replacement monotonicity,* that

$$\varphi_{k\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) \geq \varphi_{k\ell}(\tilde{R}_i, R^*_{-i}) \text{ para cada } k \in N \setminus \{j\}. \tag{3}$$

If $\varphi_{i\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) = \varphi_{i\ell}(\tilde{R}_i, R^*_{-i})$ we get the result. If not, since from Lemma 6 and Step 2, $\varphi_{i\ell}(\tilde{R}_i, R^*_{-i}) = \varphi_{i\ell}(R) = \varphi_{i\ell}(R^*_{-i,j}, R_{i,j})$, we have $\varphi_{i\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) > \varphi_{i\ell}(R^*_{-i,j}, R_{i,j})$ which implies, by *replacement monotonicity,* that

$$\varphi_{k\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) \leq \varphi_{k\ell}(R^*_{-i,j}, R_{i,j}) \text{ para cada } k \in N \setminus \{i, j\}. \tag{4}$$

Take $k \in N \setminus \{i, j\}$. By Lemma 6, Step 2 and *replacement monotonicity,* $\varphi_{k\ell}(R^*_{-i,j}, R_{i,j}) = \varphi_{k\ell}(R) = \varphi_{k\ell}(\tilde{R}_i, R^*_{-i})$. In consequence, by (3) and (4) we have $\varphi_{k\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) = \varphi_{k\ell}(\tilde{R}_i, R^*_{-i})$. By *replacement monotonicity,* $\varphi_{j\ell}(\tilde{R}_i, R^*_{-i,j}, R_j) = \varphi_{j\ell}(\tilde{R}_i, R^*_{-i})$, a contradiction. Continuing in the same fashion we can prove, for each $S \subsetneq N \setminus \{i\}$, that $\varphi_{j\ell}(\tilde{R}_i, R^*_{-S}, R_S) = \varphi_{j\ell}(\tilde{R}_i, R^*_{-i})$. We get the result considering $S = N \setminus \{i, j\}$. $\square$

Lemma 8 *Let φ be a strategy-proof, unanimous and replacement monotonic rule. Then φ is own peak-only.*

Proof. Let φ be a rule that satisfies the properties listed in the lemma. By Lemma 4, φ is *same-sided,* and by Remark 2, φ is *non-bossy.* Assume that φ is not *own peak-only.* Then there are $R \in \mathcal{R}^N$, $i \in N$, $\ell \in L$, $R'_i \in \mathcal{R}$ with $p(R'_i) = p(R_i)$ such that, without loss of generality, $\varphi_{i\ell}(R'_i, R_{-i}) < \varphi_{i\ell}(R)$. By *same-sidedness,* both $\varphi_{i\ell}(R'_i, R_{-i})$ and $\varphi_{i\ell}(R)$ are on the same side of the peak. Assume, again without loss of generality, that $p_\ell(R_i) \leq \varphi_{i\ell}(R'_i, R_{-i}) < \varphi_{i\ell}(R)$. By feasibility and *same-sidedness*, there is $j \in N \setminus \{i\}$ such that $p_\ell(R_j) \leq \varphi_{j\ell}(R'_i, R_{-i}) < \varphi_{j\ell}(R)$. Let $R^*_i \in \mathcal{R}$ be such that $p(R^*_i) = \varphi_i(R'_i, R_{-i})$ and let $R^*_j \in \mathcal{R}$ be such that $p(R^*_j) = \varphi_i(R)$. Then, it is easily seen that $\ell \in L(R_i, R^*_i, \varphi_i(R))$ and $\ell \in L(R_j, R^*_j, \varphi_j(R'_i, R_{-i}))$. By Lemma 7, $\varphi_{i\ell}(R^*_{i,j}, R_{-i,j}) = \varphi_{i\ell}(R)$, and $\varphi_{j\ell}(R^*_{i,j}, R_{-i,j}) = \varphi_{j\ell}(R'_i, R_{-i})$. By Lemma 6, $\varphi(R'_i, R_{-i}) = \varphi(R^*_i, R_{-i})$, and therefore $\varphi_{j\ell}(R^*_{i,j}, R_{-i,j}) = \varphi_{j\ell}(R^*_i, R_{-i})$, and in consequence *non-bossiness* implies $\varphi_{i\ell}(R^*_{i,j}, R_{-i,j}) = \varphi_{i\ell}(R^*_i, R_{-i}) = \varphi_{i\ell}(R'_i, R_{-i})$. It follows that $\varphi_{i\ell}(R) = \varphi_{i\ell}(R^*_{i,j}, R_{-i,j}) = \varphi_{i\ell}(R'_i, R_{-i})$. This contradicts our hypothesis. $\square$

Proof of Theorem 3. Let φ be a rule that satisfies the properties listed in the theorem. By Lemma 8, φ is *own peak-only.* Let $\psi \in \Phi^*_{SP}$ be such that ψ *dom* φ and $\psi \neq \varphi$. Then, by Lemma 3, for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$ we have $O^\psi_i(R_{-i}) \subseteq O^\varphi_i(R_{-i})$. If for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$, we have $O^\psi_i(R_{-i}) = O^\varphi_i(R_{-i})$, then by Lemma 2 (iii) we have, for each $i \in N$ and each $R \in \mathcal{R}^N$, $\psi_i(R) = C_i(R_i, O^\psi_i(R_{-i})) = C_i(R_i, O^\varphi_i(R_{-i})) = \varphi_i(R)$, and thus $\psi = \varphi$, contradicting our hypothesis. Therefore, there are $i \in N$ and $R_{-i} \in \mathcal{R}^{N \setminus \{i\}}$ such that $O^\psi_i(R_{-i}) \subsetneq O^\varphi_i(R_{-i})$. It follows that there is $x'_i \in X$ such that $x'_i \in O^\psi_i(R_{-i})$ and $x'_i \notin O^\varphi_i(R_{-i})$. Let $R'_i \in \mathcal{R}$ be such that $p(R'_i) = x'_i$. Then $x'_i = C_i(R'_i, O^\psi_i(R_{-i})) = \psi_i(R'_i, R_{-i})$. Let $y_i \equiv C_i(R'_i, O^\varphi_i(R_{-i})) = \varphi_i(R'_i, R_{-i})$. Since $x'_i \notin O^\varphi_i(R_{-i})$, $y_i \neq x'_i$. Then, there is $\ell \in L$ such that, without loss of generality, $y_{i\ell} <$

$x'_{i\ell} = p_\ell(R'_i)$. By Lemma 4, φ is *same-sided,* and since $\varphi_{i\ell}(R'_i, R_{-i}) < p_\ell(R'_i)$, this implies, for each $j \in N \setminus \{i\}$, $\varphi_{j\ell}(R'_i, R_{-i}) \leq p_\ell(R_j)$. Being $\varphi_{i\ell}(R'_i, R_{-i}) < p_\ell(R'_i) = \psi_{i\ell}(R'_i, R_{-i})$, by feasibilitythere is $j \in N \setminus \{i\}$ such that $\varphi_{j\ell}(R'_i, R_{-i}) > \psi_{j\ell}(R'_i, R_{-i})$. Thus,

$$p_\ell(R_j) \geq \varphi_{j\ell}(R'_i, R_{-i}) > \psi_{j\ell}(R'_i, R_{-i}).$$

By Lemma 5, there is $R'_j \in \mathcal{R}$ such that $p(R'_j) = p(R_j)$ and $\varphi_j(R'_i, R_{-i})\ P'_j\ \psi_j(R'_i, R_{-i})$. Since ψ is *own peak-only,* we have

$$\varphi_j(R'_i, R'_j, R_{-i,j})\ P'_j\ \psi_j(R'_i, R'_j, R_{-i,j}).$$

But this last statement contradicts that ψ *dom* φ. $\square$

The next lemma presents a property of one-dimensional rules that satisfy *strategy-proofness* and *same-sidedness.*

Lemma 9 *Let $\mathcal{R}$ be the one-dimensional single-peaked domain and let φ be a* strategy-proof *and* same-sided *rule defined on that domain. For each $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$, we have*
(i) If $p(R_i) < \varphi_i(R)$ and $p(R'_i) \leq \varphi_i(R)$, then $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$,
(ii) If $p(R_i) > \varphi_i(R)$ and $p(R'_i) \geq \varphi_i(R)$, then $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$.

Proof. Let us check (i), since (ii) is analogous. Suppose $p(R_i) < \varphi_i(R)$ and $p(R'_i) \leq \varphi_i(R)$. As φ is *same-sided,* $p(R_j) \leq \varphi_j(R)$ for each $j \in N \setminus \{i\}$. Then $p(R'_i) + \sum_{N\setminus\{i\}} p(R_j) \leq \sum_N \varphi_j(R) = \Omega$. Again by *same-sidedness,* $\varphi_i(R'_i, R_{-i}) \geq p(R'_i)$. Assume $\varphi_i(R'_i, R_{-i}) \neq \varphi_i(R)$. There are two cases to analyze:
Case 1: $\boldsymbol{\varphi_i(R'_i, R_{-i}) > \varphi_i(R)}$. Then $\varphi_i(R'_i, R_{-i}) > \varphi_i(R) \geq p(R'_i)$, which contradicts *strategy-proofness* of φ.
Case 2: $\boldsymbol{\varphi_i(R'_i, R_{-i}) < \varphi_i(R)}$. It is a well-known fact that a one-dimensional *strategy-proof* and *same-sided* rule is *own peak-only* (ver AGREGAR). Let $\tilde{R}_i \in \mathcal{R}$ be such that $p(\tilde{R}_i) = p(R_i)$ and $\varphi_i(R'_i, R_{-i})\ \tilde{P}_i\ \varphi_i(R)$. As φ is *own peak-only,* $\varphi_i(R) = \varphi_i(\tilde{R}_i, R_{-i})$, and therefore $\varphi_i(R'_i, R_{-i})\ \tilde{P}_i\ \varphi_i(\tilde{R}_i, R_{-i})$, contradicting *strategy-proofness.* $\square$

Proof of Theorem 4. Let ϕ be a multidimensional sequential rule. Notice that ϕ inherits the properties of *replacement monotonicity* and *same-sidedness* from each of its coordinate functions. We need to see that ϕ is also *strategy-proof.* Let be $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$. We must prove that $\phi_i(R)\ R_i\ \phi_i(R'_i, R_{-i})$. Take $\ell \in L$ and assume $\sum_N p_\ell(R_j) \geq \Omega_\ell$. By *same-sidedness,* $\phi_{i\ell}(R) \leq p_\ell(R_i)$. Assume $\phi_{i\ell}(R) < p_\ell(R_i)$. If $p_\ell(R'_i) \geq \phi_{i\ell}(R)$ then, by Lemma 9, $\phi_{i\ell}(R'_i, R_{-i}) = \phi_{i\ell}(R)$. If $p_\ell(R'_i) < \phi_{i\ell}(R)$, we have two cases to analyze.
Case 1: $\boldsymbol{p_\ell(R'_i) + \sum_{N\setminus\{i\}} p_\ell(R_j) \geq \Omega_\ell}$. By *same-sidedness,* $\phi_{i\ell}(R'_i, R_{-i}) \leq p_\ell(R'_i) < \phi_{i\ell}(R)$.
Case 2: $\boldsymbol{p_\ell(R'_i) + \sum_{N\setminus\{i\}} p_\ell(R_j) < \Omega_\ell}$. Assume $\phi_{i\ell}(R'_i, R_{-i}) > \phi_{i\ell}(R)$. By feasibility there is $j \in N \setminus \{i\}$ such that $\phi_{j\ell}(R'_i, R_{-i}) < \phi_{j\ell}(R)$. By *same-sidedness* we have $p_\ell(R_j) \leq \phi_{j\ell}(R'_i, R_{-i}) < \phi_{j\ell}(R) \leq p_\ell(R_j)$, which is absurd. In consequence, $\phi_{i\ell}(R'_i, R_{-i}) \leq \phi_{i\ell}(R)$.

We conclude that, either $\phi_{i\ell}(R) = p_\ell(R_i)$ or $\phi_{i\ell}(R'_i, R_{-i}) \leq \phi_{i\ell}(R) < p_\ell(R_i)$. With an analogous reasoning we can see that, for $\ell \in L$ such that $\sum_N p_\ell(R_j) < \Omega_\ell$, we have $\phi_{i\ell}(R) = p_\ell(R_i)$ ó $\phi_{i\ell}(R'_i, R_{-i}) \geq \phi_{i\ell}(R) > p_\ell(R_i)$. This implies $\phi_i(R) R_i \phi_i(R'_i, R_{-i})$. $\square$

Proof of Theorem 5. The multidimensional uniform rule is *strongly Pareto dominant strategy-proof* by Corollary 2 and, therefore, *Pareto dominant strategy-proof.* Moreover, it satisfies *equal treatment.* We already mentioned that it is *unanimous* and *replacement monotonic.* Let φ be a rule that satisfies the properties listed in the theorem. Then, in particular, φ is *strategy-proof, unanimous, equally-treating* and *non-bossy.* It follows, from Corollary 1 in Morimoto, Serizawa and Ching, that $\varphi = U$. □

References

[1] Adachi, T. (2010). "The uniform rule with several commodities: a generalization of Sprumont's characterization". *Journal of Mathematical Economics* 46, 952-964.

[2] Amorós, P. (1999). "Efficiency and income redistribution in the single-peaked preferences model with several commodities". *Economics Letters* 63, 341-349.

[3] Amorós, P. (2002). "Single-peaked preferences with several commodities". *Social Choice and Welfare* 19, 57-67.

[4] Anno, H. y H. Sasaki. (2013). "Second-best efficiency of allocation rules: strategy-proofness and single-peaked preferences with multiple commodities". *Economic Theory* 54, 693-716.

[5] Barberà, S. y M. Jackson. (1995). "Strategy- proof exchange". *Econometrica* 63, 51-87.

[6] Barberà, S., M. Jackson y A. Neme. (1997). "Strategy-proof allotment rules". *Games and Economic Behavior* 18, 1-21.

[7] Bénassy, J-P. (1982). *The Economics of Market Disequilibrium.* New York: Academic Press.

[8] Cho, W. (2014). "Impossibility results for parametrized notions of efficiency and strategy-proofness in exchange economies". *Games and Economic Behavior* 86, 26-39.

[9] Cho, W. y W. Thomson. (2013). "Strategy-proofness in private good economies with linear preferences: an impossibility result". Working paper, University of Rochester.

[10] Dasgupta, P., P. Hammond y E. Maskin. (1979). "The implementation of social choice rules: some general results on incentive compatibility". *Review of Economic Studies* 46, 185-216.

[11] Gibbard, A. (1973). "Manipulation of voting schemes: a general result". *Econometrica* 41, 587-602.

[12] Goswami, M. P., M. Mitra y A. Sen. (2014). "Strategy-proofness and Pareto-efficiency in quasi-linear exchange economies". *Theoretical Economics* 9, 361-381.

[13] Hurwicz, L. (1972) . “On informationally decentralized systems” in McGuire, C. B. and R. Radner (eds.) *Decision and Organization* (North-Holland), 297-336.

[14] Ju, B.-G. (2003). “Strategy-proofness versus efficiency in exchange economies: general domain properties and applications”. *Social Choice and Welfare* 21, 73-93.

[15] Momi, T. (2013). “Note on social choice allocation in exchange economies with many agents”. *Journal of Economic Theory* 148, 1237-1254.

[16] Morimoto, S., S. Serizawa y S. Ching. (2013). “A characterization of the uniform rule with several commodities and agents”. *Social Choice and Welfare* 40, 871-911.

[17] Moulin, H. (1980). “On Strategy-proofness and single-peakness”. *Public Choice* 35, 437-455.

[18] Nicoló, A. (2004). “Efficiency and truthfulness with Leontief preferences. A note on two-agent, two-good economies”. *Review of Economic Design* 8, 373-382.

[19] Satterthwaite, M. (1975). “Strategy-proofness and Arrow’s conditions: existence and correspondence theorems for voting procedures and social welfare functions”. *Journal of Economic Theory* 10, 187-217.

[20] Schummer, J. (1997). “Strategy-proofness versus efficiency on restricted domains of exchange economies”. *Social Choice and Welfare* 14, 47-56.

[21] Serizawa, S. (2002). “Inefficiency of strategy-proof rules for pure exchange economies”. *Journal of Economic Theory* 106, 219-241.

[22] Sprumont, Y. (1991). “The division problem with singled-peaked preferences: a characterization of the uniform allocation rule”. *Econometrica*, 59, 509-519.

[23] Zhou, L. (1991). “Inefficiency of strategy-proof allocation mechanisms in pure exchange economies”. *Social Choice and Welfare* 8, 247-254.