Social Choice Situations and Aggregation Criteria:

Inductive Reasoning in Social Choice Theory

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Abstract

The usual procedure in Social Choice Theory consists in postulating some desirable properties which a aggregation procedure should verify and from them to derive the features of the corresponding social choice function and the outcomes that arise at each possible profile of preferences. In this paper we invert this line of reasoning and try to infer, up from what we call *social situations* (each one consisting of a profile and the associated social ordering) the criteria verified in the implicit aggregation procedure. Furthermore we derive them in axiomatic form. This inference process, which extracts intentional from extensional information can be seen as an exercise in social choice-theoretic "statistics". The fact that complete intentional characterizations of the aggregation process cannot be derived in such way can be easily seen as a consequence of the procedure.

Keywords: Social Choice, Aggregation, Model Theory, Axiomatization.
J.E.L. Codes: C00, D71.

Resumen

El procedimiento usual en la Teoría de la Elección Social consiste en postular algunas propiedades que se desea que un proceso de agregación verifique y de ellas derivar las características de la función de agregación social y los resultados que surgen de aplicarla en cada posible vector de preferencias. En este trabajo invertimos esta línea de razonamiento y tratamos de inferir, a partir de lo que llamamos *situaciones sociales* (cada una consistente de un vector de preferencias y del ordenamiento social asociado), los criterios verificados en el proceso de agregación implícito. Más aún, los derivamos en forma axiomática. Este proceso de inferencia, que extrae información intensional a partir de la extensional puede ser visto como un ejercicio en "estadística" de la elección social. El hecho de caracterizaciones intensionales completas no pueden ser derivadas en esta forma puede ser fácilmente visto como una consecuencia del procedimiento.
Palabras Clave: Elección Social, Agregación, Teoría de Modelos, Axiomatización.

1 Introduction

Social outcomes arise from rather complicated decision-making arrangements among individuals and organizations. The study of these processes of *aggregation* of individual opinions to yield collective decisions gained momentum in the late 1940s. K.Arrow proved that a very simple set of conditions on the aggregation process, intended to represent the features of democratic systems, were impossible to be fulfilled simultaneously. Since then a great number of conditions both limiting and expanding the possibilities of aggregation have been found. Moreover, these results have been shown to be relevant for other decision-making settings–like the design of institutional mechanisms.

In this paper we will invert the usual procedure in Social Choice Theory. Instead of postulating conditions for the aggregation processes we will define the class of their possible results. Then we will look for a partition of this class according to the implicit aggregation procedures that yield them. We will look then for the necessary conditions verified by each equivalence class. This means that instead of characterizing results in terms of the conditions on the aggregation procedure, we will characterize the conditions based on the results they yield.

This procedure, which consists in deriving an *intensional* characterization up from an *extensional* one is typical in statistics. This is why we think of this paper as an exercise in social choice-theoretic statistics. As we will see, there is no complete equivalence between intensional and extensional characterizations of social choice functions. Even so, some partial information can be drawn by means of this procedure. At the interpretation level, it is interesting to note that, while social choice functions can be seen as the determination of a *constitution*, our approach detects an underlying and more general context in which there exist some kind of *partial agreements* that set the ground for possible future constitutions.

As it is obvious, any induction process depends on a previous criterion of relevance, which defines what features should be taken into account as well as which others should be dismissed. In the specific exercise we develop here, we focus on the idea that names of agents and of alternatives are irrelevant; in other words, we are dealing with the general notions that give place to the well-known conditions of *neutrality* and *anonymity*. But our approach does not merely stipulate those conditions, they just arise in the determination

of equivalence classes of social choice situations.[1]

In addition, those equivalences are defined by means of the decision structure associated with each social choice situation. Actually, we can speak of the decision structure of a situation only in a derivative way since decision structures themselves arise as the result of the equivalence between social choice situations. Thus, those situations that are equivalent exhibit certain similarities, particularly with respect to the decisive sets in each situation. Finally, given our definition of equivalence, there will exist different degrees of similarity depending on which permutations (over agents, and over alternatives) are admissible. Thus, our approach allows at least two levels of generalization: not only we consider notions contained in typical social choice conditions (as neutrality, and anonymity) in a wider perspective, but also we allow them to exhibit a certain variability.

After obtaining the classification of social choice situations terms of equivalence classes, we use this information to connect them with a more typical element in Social Choice theory, aggregation functions. The main part of this paper studies this connection. Finally, once we have associated conditions with aggregation functions we look for axiomatizations of the aggregation procedures. Notices that this obtains up from the previous classification of social choice situations. Since they obtain as necessary conditions, some information is lost. But this is, as said, typical in the extraction of intensional information from extensional descriptions. Even so, it is interesting to see that the basic features of aggregation functions (their *partial agreements* aspects) can be finitely axiomatized. This yields a partial converse to the typical procedure in Social Choice.

The framework for this study is well known in the field of Social Choice Theory: the theory of models. In fact, since the mathematical analysis of axiom systems relies heavily on the structures that satisfy them (their *models*)it is natural to consider that most of this theory is of model-theoretic nature. Beyond this immediate use of model-theoretic notions, a more sophisticated toolbox has been applied in the field. Some of those notions will be used in our derivation of the axiomatizations of aggregation functions up from social situations.

The plan of the paper is as follows. In section 2 we present the partition of social choice situation in terms of permutations of the names of both agents and situations. In section 3 we derive the aggregation functions

[1] A related distinction, between criteria and axioms, is developed by Campbell & Kelly (1997).

that correspond to the equivalence classes found in the previous section. In section 4 we relate the aggregation functions with social choice criteria (verified by the elements in the corresponding equivalence class). In section 5 we derive axiomatizations of the aggregation procedures, in 5.1 for finite social choice situations and in 5.2 for infinite ones. Finally, section 6 concludes.

2 Partitions of the Space of Social Choice Situations

We will define a *social choice situation* with n individuals over a countable set of alternatives S ($|S| \geq 3$) as $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R} \rangle$, where each $R_i \subseteq S \times S$ is agent i's preferential ordering and $\bar{R} \subseteq S \times S$ is the *social* ordering of S. We restrict our attention to the situations in which each individual ordering, $R \in \{R_1, \ldots, R_n\}$, is:

- *reflexive*: $R(s, s)$ for each $s \in S$.
- *complete*: $R(s, t)$ or $R(t, s)$ for every $s, t \in S$.
- *transitive*: if $R(s, t)$ and $R(t, u)$ then $R(s, u)$ for $s, t, u \in S$.

These conditions, that define R as a *weak order*, are usually seen as representing the rationality (consistency) of a decision-maker. The class of situations described so far is denoted $\widehat{\mathcal{S}}$.

We will assume that the expression *social choice situation* applies only to the profiles that verify these conditions. Therefore, any classification of social choice situation will implicitly characterize either existential or universal conditions. As we wish to classify situations and *then* extract the information of the conditions verified by the social choice situations, we should place ourselves in the position of an outsider observer who has to decide which features must deemed relevant and which accessory.

The first thing to note is that the *names* of both agents and alternatives may be irrelevant. The families of admissible automorphisms ("name changes") over S and $\{1, \ldots, n\}$ may have any structure, ranging from being singletons to the full blown group of permutations. However, to keep a certain symmetry we will require them to have an **group** structure. That is, if $\mathcal{G} = \{f : A \to A\}$ represents either the automorphisms allowed on the set of alternatives ($A = S$) or on the individuals ($A = \{1, \ldots, n\}$) the following conditions must be verified:

- If $f, g \in \mathcal{G}$, $f \circ g \in \mathcal{G}$, where $\circ$ is the composition of automorphisms.
- $Id \in \mathcal{G}$, i.e. the identity (which leaves the names invariant) must always be included in the family of admissible automorphisms,
- for every $f \in \mathcal{G}$ there must exist $f^{-1} \in \mathcal{G}$ such that $f \circ f^{-1} = Id = f^{-1} \circ f$. That is, for each automorphism which is admissible, its inverse should also be admissible.

As a trivial consequence of Cayley's theorem, the group of automorphisms over a finite set A coincides with the group of permutations of $|A|$, denoted $\mathbf{S}_{|A|}$. We will fix generic admissible permutation groups over S and $\{1, \ldots, n\}$, $\mathcal{G}_S$ and $\mathcal{G}_n$, respectively, such that $\mathcal{G}_S$ is a subgroup of $\mathbf{S}_{|S|}$ and $\mathcal{G}_n$ a subgroup of $\mathbf{S}_n$.

We will consider the social situations that are equivalent under the admissible permutations. That is, we consider a binary relation on $\widehat{\mathcal{S}}$, $\simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)}$, which we call *decision-equivalence*:

Definition 1 *Two social choice situations, $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R} \rangle$ and $\mathcal{S}' = \langle S, R_1', \ldots, R_n'; \bar{R}' \rangle$ are decision-equivalent, $\mathcal{S} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}'$, if and only there exist two permutations $\rho \in \mathcal{G}_n$ and $\gamma \in \mathcal{G}_S$ such that for each pair $s, t \in S$ we have for $D^{\mathcal{S}}(\{s,t\}) = \{i : R_i(s,t)$ iff $\bar{R}(s,t)\}$ and $D^{\mathcal{S}'}(\{\gamma(s), \gamma(t)\}) = \{i : R_i'(\gamma(s), \gamma(t))$ iff $\bar{R}'(\gamma(s), \gamma(t))\}$ that*

$$\rho(D^{\mathcal{S}}(\{s,t\})) = \{\rho(i) : i \in D^{\mathcal{S}}(\{s,t\})\} = D^{\mathcal{S}'}(\{\gamma(s), \gamma(t)\})$$

or[2]*, if $\bar{R}(s,t)$ and $\bar{R}'(\gamma(s), \gamma(t))$ are* both *undefined, we state by convention that $D^{\mathcal{S}}(\{s,t\}) = \emptyset = D^{\mathcal{S}'}(\{\gamma(s), \gamma(t)\})$.*

In words: two social choice situations are decision-equivalent if and only if there exist changes in the names of both agents and alternatives that make them coincide. Notice that a crucial element in this definition is the characterization of the *decisive sets* over pairs of alternatives, i.e. given any two alternatives, the set of agents that coincide with the social preferences over them.

Different families $\mathcal{G}_S$ and $\mathcal{G}_n$ yield alternative decision-equivalence relations. The particular choice of those groups of permutations implies the degrees of *neutrality* and *anonymity* that are to be captured by the equivalence

[2]Recall that we do not place any restriction on the social order. We do not ask, in particular, it to be complete.

relation. Notice that certain social situations may be seen as equivalent, independently of the admissible permutations:

Example 1 *Consider two situations in which three agents (*$1, 2$ *and* 3*) have to decide:*

$$\mathcal{S} = \langle \{a, b, c\}, \{a \prec b \prec c\}_1, \{b \prec c \prec a\}_2, \{a \prec b \prec c\}_3; \{\{b \prec c\}, \{a\}\}\rangle$$

and

$$\mathcal{S}^{'} = \langle \{a, b, c\}, \{b \prec c \prec a\}_1, \{b \prec a \prec c\}_2, \{a \prec b \prec c\}_3; \{\{b \prec c\}, \{a\}\}\rangle$$

In both situations, the agents order linearly the three alternatives a, b, c *and the social outcome is the same:* c *is preferred to* b *while* a *does not bear any relation with the other two alternatives. Since all three agents,* $1, 2$ *and* 3 *prefer* c *to* b *we have that*

$$D^{\mathcal{S}}(\{b, c\}) = \{1, 2, 3\} = D^{\mathcal{S}^{'}}(\{b, c\})$$

while, since a *is not related in the social orderings to* b *and* c *we have that*

$$D^{\mathcal{S}}(\{b, a\}) = D^{\mathcal{S}}(\{a, c\}) = \emptyset = D^{\mathcal{S}^{'}}(\{b, a\}) = D^{\mathcal{S}^{'}}(\{a, c\})$$

That is, $\mathcal{S} \simeq_{\mathcal{D}} \mathcal{S}^{'}$ *under* Id_n *and* Id_S*, the identity permutations over* $\{1, \ldots, n\}$ *and* S *respectively.*

We have that

Proposition 1 *The class* $\widehat{\mathcal{S}}$ *of social choice situations is partitioned by* $\simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)}$.

Proof of Proposition 1 *It is immediate from the definition of* $\simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)}$ *that for any situation* $\mathcal{S}$*,* $\mathcal{S} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}$*, just taking the identity permutations* Id_n *and* Id_S*. That is,* $\simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)}$ *is reflexive. It is also symmetric: if* $\mathcal{S} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}^{'}$ *under a* $\rho \in \mathcal{G}_n$ *and a* $\gamma \in \mathcal{G}_S$*, then* $\mathcal{S}^{'} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}$*, just taking* $\rho^{-1} \in \mathcal{G}_n$ *and* $\gamma^{-1} \in \mathcal{G}_S$*, which are admissible since* $\mathcal{G}_n$ *and* $\mathcal{G}_S$ *are groups. On the other hand, when* $\mathcal{S} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}^{'}$ *and* $\mathcal{S}^{'} \simeq_{\mathcal{D}(\mathcal{G}_n, \mathcal{G}_S)} \mathcal{S}^{''}$ *we have that there exist permutations,* $\rho, \rho^{'} \in \mathcal{G}_n$ *and* $\gamma, \gamma^{'} \in \mathcal{G}_S$ *such that for each* $s, t \in S$

$$\rho(D^{\mathcal{S}}(\{s, t\})) = D^{\mathcal{S}^{'}}(\{\gamma(s), \gamma(t)\}) \text{ and}$$

$$\rho^{'}(D^{\mathcal{S}^{'}}(\{\gamma(s),\gamma(t)\})) = D^{\mathcal{S}^{''}}(\{\gamma^{'}(s),\gamma^{'}(t)\})$$

That is

$$\rho^{'} \circ \rho(D^{\mathcal{S}}(\{s,t\})) = D^{\mathcal{S}^{''}}(\{\gamma^{'} \circ \gamma(s),\gamma^{'} \circ \gamma(t)\})$$

Since $\rho^{'} \circ \rho \in \mathcal{G}_n$ *and* $\gamma^{'} \circ \gamma \in \mathcal{G}_S$, $\mathcal{S} \simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)} \mathcal{S}^{''}$. *That is,* $\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *is transitive.* □

Over each equivalence class $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ we can distinguish a particular class of decisive sets which will allow us to identify the class:

Definition 2 *For* $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *the class, up to permutations, of its* decision sets *is* $\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$ *such that for each situation* $\mathcal{S} \in \bar{\mathcal{S}}$ *there exist* $\rho \in \mathcal{G}_n$ *and* $\gamma \in \mathcal{G}_S$ *such that for each pair* $s,t \in S$, $D(\{s,t\}) \in \mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$, $D(\{s,t\}) = \rho(D^{\mathcal{S}}(\{\gamma^{-1}(s),\gamma^{-1}(t)\}))$.[3]

With these definitions at hand we can now study the structure of *aggregation* functions:

Definition 3 *Given any situation* $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R}\rangle$, *its associated* preference profile *is* $prof(\mathcal{S}) = \langle R_1, \ldots, R_n\rangle$. *For the entire class of situations,* $\widehat{\mathcal{S}}$, *we associate the class* $prof(\widehat{\mathcal{S}}) = \{prof(\mathcal{S}) : \mathcal{S} \in \bigcup\widehat{\mathcal{S}}\}$ *of all possible preference profiles. If* $Ord(S)$ *denotes the class of all possible orderings of* S, *a (partial)* aggregation function *is*

$$\mathbf{f} : prof(\widehat{\mathcal{S}}) \rightarrow Ord(S)$$

In words: an aggregation function takes as its argument a profile of preferences over S, $\text{prof}_{\mathcal{S}}$, and yields a (partial) ordering of S.[4] A further refinement that will be shown useful in the following is:

Definition 4

- *Given the permutation groups* $\mathcal{G}_n$ *and* $\mathcal{G}_S$, $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *is the class of aggregation functions such that, given* $\rho \in \mathcal{G}_n$ *and* $\gamma \in \mathcal{G}_S$, *each* $\mathbf{f} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *verifies* $\gamma(\mathbf{f}(prof_{\mathcal{S}})) = \mathbf{f}(\rho(prof)_{\gamma(\mathcal{S})})$ *where* $prof_{\mathcal{S}} = \langle R_1, \ldots, R_n\rangle$ *over* S *and* $\rho(prof)_{\gamma(\mathcal{S})} = \langle R_{\rho(1)}, \ldots, R_{\rho(n)}\rangle$ *over* $\gamma(S)$, *while* $\gamma(R)(s,t) = R(\gamma(s),\gamma(t))$ *for* $s,t \in S$.

[3] In more technical terms this means that for each $\mathcal{S} \in \bar{\mathcal{S}}$ there exist two permutations, ρ and γ, such that $\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$ is equivalent *modulo* ρ,γ, to the class of decision sets of $\mathcal{S}$.

[4] Notice that, on one hand, $\mathbf{f}(\text{prof}_{\mathcal{S}})$ will not necessarily coincide with the social relation R in the situation $\mathcal{S}$. On the other hand, there might exist at least two elements $s,t \in S$ such that $\mathbf{f}(\text{prof}_{\mathcal{S}})(s,t) = \emptyset$.

- *An aggregation function* $\mathbf{f} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *is said* prime *iff*
  - *for every profile* $prof_{\mathcal{S}}$, $\mathbf{f}(prof_{\mathcal{S}}) = \emptyset$

  or

  - *there does not exist another aggregation function* $\mathbf{f}' \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *verifying that for* $prof_{\mathcal{S}}$ *such that* $\mathbf{f}(prof_{\mathcal{S}}) \neq \emptyset$, $\emptyset \neq \mathbf{f}'(prof_{\mathcal{S}}) \subseteq \mathbf{f}(prof_{\mathcal{S}})$ *while* $\mathbf{f}'(prof_{\mathcal{S}'}) = \emptyset$ *for every profile* $prof_{\mathcal{S}'}$ *such that* $\mathbf{f}(prof_{\mathcal{S}'}) = \emptyset$.

The domain of an aggregation function $\mathbf{f} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ is $\mathrm{Dom}(\mathbf{f}) = \{\mathrm{prof}_{\mathcal{S}} \in \mathrm{prof}(\widehat{\mathcal{S}}) : \mathbf{f}(\mathrm{prof}_{\mathcal{S}}) = \mathbf{f}(\rho(\mathrm{prof})_{\gamma(\mathcal{S})}) \neq \emptyset, \quad \rho \in \mathcal{G}_n, \gamma \in \mathcal{G}_S\}$ while its range is $\mathrm{Im}(\mathbf{f}) = \{R \in \mathrm{Ord}(S) : \exists \mathrm{prof}_{\mathcal{S}} \in \mathrm{prof}(\widehat{\mathcal{S}}), \mathbf{f}(\mathrm{prof}_{\mathcal{S}}) = \mathbf{f}(\rho(\mathrm{prof})_{\gamma(\mathcal{S})}) = R, \quad \rho \in \mathcal{G}_n, \gamma \in \mathcal{G}_S\} \cup \emptyset$. Since each $\mathbf{f} = \mathrm{Dom}(\mathbf{f}) \times \mathrm{Im}(\mathbf{f})$, $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ may be ordered by $\subseteq \times \subseteq$. This ordering of aggregation functions reflects the corresponding relation among decision sets:

Example 2 *Consider two situations:*

$$\mathcal{S} = \langle \{a,b,c\}, \{a \prec b \prec c\}_1, \{b \prec c \prec a\}_2, \{c \prec a \prec b\}_3; \{a \prec b \prec c \prec a\} \rangle$$

and

$$\mathcal{S}' = \langle \{a,b,c\}, \{a \prec b \prec c\}_1, \{a \prec b \prec c\}_2, \{a \prec b \prec c\}_3; \{a \prec b \prec c\} \rangle$$

These two situations are not equivalent since, for example, $D^{\mathcal{S}}(\{b,c\}) = \{1,2\}$ *while* $D^{\mathcal{S}'}(\{b,c\}) = \{1,2,3\}$ *and there do not exist two possible permutations* ρ *and* γ *such that* $\rho(D^{\mathcal{S}}(\{b,c\})) = D^{\mathcal{S}'}(\{\gamma(b),\gamma(c)\})$. *Consider an aggregation function* $\mathbf{f}$ *such that*

$$\{a \prec b \prec c \prec a\} = \mathbf{f}(\{a \prec b \prec c\}_1, \{b \prec c \prec a\}_2, \{c \prec a \prec b\}_3)$$

and

$$\{a \prec b \prec c\} = \mathbf{f}(\{a \prec b \prec c\}_1, \{a \prec b \prec c\}_2, \{a \prec b \prec c\}_3)$$

A possibility is to consider that $\mathbf{f}$ *is the* majority rule *which, for every pair of alternatives, yields the order among them preferred by the majority of the agents (two over three, in this case). But this aggregation function is not prime since for example the* unanimous rule *that yields only the order that is agreed upon by* all *the agents can explain also the social outcome in situation* $\mathcal{S}'$.

This example shows, in particular, that non-equivalent situations may respond to different aggregation functions (if one at least is not prime). This intuition becomes useful in the following:

Lemma 1 *Given $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ there exists a prime aggregation function $\mathbf{f}_{\bar{\mathcal{S}}} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ such that for any situation $\mathcal{S} = \langle S, prof_{\mathcal{S}}; \bar{R}\rangle \in \bar{\mathcal{S}}$, $\mathbf{f}_{\bar{\mathcal{S}}}(prof_{\mathcal{S}}) = \bar{R}$. Conversely, for any prime aggregation function $\mathbf{f} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ there exists a $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ such that $\mathbf{f}(prof_{\mathcal{S}}) = \bar{R} \neq \emptyset$ for every $\mathcal{S} = \langle S, prof_{\mathcal{S}}; \bar{R}\rangle \in \bar{\mathcal{S}}$.*

Proof of Lemma 1 $\Rightarrow$) *Given $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ consider the family of its decision sets, $\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$ and define a function $\mathbf{f}_{\bar{\mathcal{S}}}$ as follows. For each profile $\langle R_1, \ldots, R_n\rangle$ and for each pair $s, t \in S$, consider the decisive set $D(\{s,t\}) \in \mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$. Then, for every $\rho \in \mathcal{G}_n$ and $\gamma \in \mathcal{G}_S$:*

$$\mathbf{f}(\rho(prof)_{\gamma(\mathcal{S})}) = R(\gamma(s), \gamma(t)) \text{ iff for every } i \in D(\{s,t\}), R_{\rho(i)}(\gamma(s), \gamma(t)).$$

By the definition of $\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}})$, $\mathbf{f}_{\bar{\mathcal{S}}} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$. Moreover, if for every $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R}\rangle \notin \bar{\mathcal{S}}$ we define $\mathbf{f}_{\bar{\mathcal{S}}}(R_1, \ldots, R_n) = \emptyset$, it is immediate that $\mathbf{f}_{\bar{\mathcal{S}}}$ is prime.
$\Leftarrow$) *Consider the class of situations with nonempty social orders which can be seen as defined by an arbitrary prime aggregation function $\mathbf{f} \in \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$, $\widehat{\mathcal{S}}^{\mathbf{f}} = \{\mathcal{S} \in \widehat{\mathcal{S}} : \mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R}\rangle, \mathbf{f}(R_1, \ldots, R_n) = \bar{R} \neq \emptyset\}$. Assume, by way of contradiction, that there exist two equivalence classes $\bar{\mathcal{S}}, \bar{\mathcal{S}}' \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ such that $\widehat{\mathcal{S}}^{\mathbf{f}} \cap \bar{\mathcal{S}} \neq \emptyset$ and $\widehat{\mathcal{S}}^{\mathbf{f}} \cap \bar{\mathcal{S}}' \neq \emptyset$. Consider two situations $\mathcal{S} \in \widehat{\mathcal{S}}^{\mathbf{f}} \cap \bar{\mathcal{S}}$ and $\mathcal{S}' \in \widehat{\mathcal{S}}^{\mathbf{f}} \cap \bar{\mathcal{S}}'$. Then, $\mathcal{S}$ and $\mathcal{S}'$ are not equivalent. This can happen for two reasons: for every pair of permutations ρ and γ there exists at least a pair $s, t \in S$ such that either*

- $D^{\bar{\mathcal{S}}}(\{s,t\}) \neq \rho(D^{\bar{\mathcal{S}}'}(\{\gamma^{-1}(s), \gamma^{-1}(t)\}))$

or

- *either $\bar{R}(s,t)$ or $\bar{R}'(\gamma(s), \gamma(t))$ is not defined.*

Let us analyze the latter case. Without loss of generality, assume that $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R}\rangle$ and $\mathcal{S}' = \langle S, R_1', \ldots, R_n'; \bar{R}'\rangle$ are such that $R_i \equiv R_i'$ for $i = 1, \ldots, n$ while $\bar{R}(s,t)$ and not $\bar{R}'(\gamma(s), \gamma(t))$ (nor $\bar{R}'(\gamma(t), \gamma(s))$), in particular for $\gamma \equiv Id_S$. Since $\mathcal{S}, \mathcal{S}' \in \widehat{\mathcal{S}}^{\mathbf{f}}$, $\bar{R} = \mathbf{f}(R_1, \ldots, R_n)$ and $\bar{R}' = \mathbf{f}(R_1, \ldots, R_n)$. But then, since we assumed that $\bar{R} \not\equiv \bar{R}'$ we have a contradiction because $\mathbf{f}$ is assumed to be a function.

Now suppose that $D^{\bar{\mathcal{S}}}(\{s,t\}) \neq \rho(D^{\bar{\mathcal{S}}'}(\{\gamma^{-1}(s),\gamma^{-1}(t)\}))$ *for at least a pair of alternatives,* s *and* t*. According to the* $\Rightarrow$ *part of this proof there exist two prime aggregation functions* $\mathbf{f}'$ *and* $\mathbf{f}''$ *in* $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *such that* $\mathbf{f}'(R_1,\ldots,R_n) = \bar{R}$ *and* $\mathbf{f}''(R_1,\ldots,R_n) = \bar{R}'$*. On the other hand we have that* $\mathbf{f}'(R_1,\ldots,R_n) \subseteq \mathbf{f}(R_1,\ldots,R_n)$ *and* $\mathbf{f}''(R_1,\ldots,R_n) \subseteq \mathbf{f}(R_1,\ldots,R_n)$*, but this is absurd since* $\mathbf{f}$ *is prime. This shows that there exists an equivalence class* $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$*, such that* $\widehat{\mathcal{S}}^{\mathbf{f}} \subseteq \bar{\mathcal{S}}$*.*
Now assume that there exists a situation $\mathcal{S} \in \bar{\mathcal{S}}$ *and* $\mathcal{S} \notin \widehat{\mathcal{S}}^{\mathbf{f}}$*. This means that if* $\mathcal{S}$ *is* $\langle S, R_1,\ldots,R_n;\bar{R}\rangle$*, with* $\bar{R} \not\equiv \mathbf{f}(R_1,\ldots,R_n)$*, there must exist a situation* $\mathcal{S}' = \langle S, R_1,\ldots,R_n;\mathbf{f}(R_1,\ldots,R_n)\rangle \in \widehat{\mathcal{S}}^{\mathbf{f}}$*. But then, for every pair of permutations* ρ *and* γ*, there must exist a least a pair* $s,t \in S$ *such that* $D^{\mathcal{S}}(\{s,t\}) \neq \rho(D^{\mathcal{S}'}(\{\gamma^{-1}(s),\gamma^{-1}(t)\}))$*. Absurd, since* $\mathcal{S}$ *and* $\mathcal{S}'$ *are both in* $\bar{\mathcal{S}}$*. Therefore* $\widehat{\mathcal{S}}^{\mathbf{f}} = \bar{\mathcal{S}}$*.* $\square$

3 Aggregation Functions and Partitions

The relation among equivalence classes in $\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ and prime functions can be extended to all aggregation functions in $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$, according to the following property:

Definition 5 *The class of aggregation functions* $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ *is partially ordered by the set-theoretical relation of inclusion among their ranges,* $\subseteq$*. That is, we say that* $\mathbf{f} \preceq \mathbf{f}'$ *iff for every profile* $prof_{\mathcal{S}}$*,* $\mathbf{f}(prof_{\mathcal{S}}) \subseteq \mathbf{f}'(prof_{\mathcal{S}})$*.*

The order $\preceq$ is indeed partial since it is reflexive, antisymmetric and transitive. It is not complete, that is, aggregation functions are not necessarily comparable. On the other hand, since we have not placed any restriction on the cardinality of S nor on the type of allowable social orderings, $\mathrm{Ord}(S)$ can be as big as $2^{|S|}$. In case S is countably infinite we are lead to a class with a cardinality highly dependent on the underlying set theory to which we adhere.[5] What really matters for our argument is that for every possible $\mathbf{f}$, its image is such that $\mathrm{Im}(\mathbf{f}) \subseteq \mathrm{Ord}(S)$. That is, $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ is bounded from above. According to *Zorn's Lemma* we have that every chain (linear suborder) in $\langle \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}, \preceq\rangle$ has a maximal element. A similar argument shows

[5] This is a consequence of the fact that the statement $2^{\aleph_0} = \aleph_1$, the so called *Continuum Hypothesis*, is independent of the ZFC (Zermelo-Fraenkel plus the Axiom of Choice) axiomatization of Set Theory. Additional axioms can force the hypothesis to be either true or false.

that, since $\emptyset$ bounds $\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ from below, the existence of minimal (*prime*) aggregation functions is ensured.

Example 3 *Let us consider the family of $q-$ quota aggregation functions: for each profile $\langle R_1,\ldots,R_n\rangle$ and each $s,t\in S$, $\mathbf{f}^q(\langle R_1,\ldots,R_n\rangle)_{(s,t)}$ iff there exists a subset $I^q\subseteq\{1,\ldots,n\}$, such that*[6] *$|I^q|=\lceil\frac{q\times n}{100}\rceil$ and for each $i\in I^q$, $R_i(s,t)$. For the decreasing sequence $q\downarrow_0^{100}$ we have the following chain:*

$$\mathbf{f}^{100}\preceq\ldots\mathbf{f}^{99}\preceq\ldots\mathbf{f}^{0}$$

i.e. from the unanimous *aggregation function to the "everything goes" function.*

The ordering among aggregation functions is reflected in the structure of social situations:

Proposition 2 *For any aggregation function $\mathbf{f}\in\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ there exists a family of equivalence classes $\{\bar{\mathcal{S}}^j\}_{j\in J}\subseteq\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ such that $\mathbf{f}(prof_{\mathcal{S}})=\bar{R}$ for every $\mathcal{S}=\langle S,prof_{\mathcal{S}};\bar{R}\rangle\in\cup_{j\in J}\bar{\mathcal{S}}^j$. $\cup_{j\in J}\bar{\mathcal{S}}^j$ is called the* support *of $\mathbf{f}$, $\mathcal{SUPP}(\mathbf{f})$.*

Proof of Proposition 2 *For every aggregation function $\mathbf{f}$ there exists a family $\{\mathbf{f}^j\}_{j\in J_{-0}}$ of prime aggregation functions such that each $\mathbf{f}^j\preceq\mathbf{f}$. Consider each $\bar{R}\in Im(\mathbf{f})$. If $\bar{R}\neq\emptyset$ just consider the equivalence classes associated to each prime function according to Lemma 1: $\{\bar{\mathcal{S}}^j\}_{j\in J_{-0}}\subseteq\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$. It is clear that for each $prof_{\mathcal{S}}$ such that $\mathcal{S}\in\cup_{j\in J_{-0}}\bar{\mathcal{S}}^j$ we have that there exists at least a $j\in J_{-0}$ that verifies that $\mathbf{f}^j(prof_{\mathcal{S}})=\bar{R}$. On the other hand, if $R=\emptyset$ then consider $\bar{\mathcal{S}}^0$ the class of all the situations of the form $\langle S,R_1,\ldots,R_n;\emptyset\rangle$. This class corresponds to the prime function $\mathbf{f}^0$ such that $\mathbf{f}^0(prof_{\mathcal{S}})=\emptyset$. Then, if we consider the class $J=J_{-0}\cup\{0\}$, we have that $\mathbf{f}(prof_{\mathcal{S}})=\bar{R}$ for every $\mathcal{S}\in\cup_{j\in J}\bar{\mathcal{S}}^j$.*$\square$

This indicates that each $\mathbf{f}\in\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}$ could be associated to a class in the *coarsening* of the partition $\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$. This shows, in turn, that the *properties* of any aggregation function can be analyzed in terms of the behavior of unions of equivalence classes in the partition $\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$.

The features of $\langle\mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S},\preceq\rangle$ are reflected in the *aggregational ordering* of $\widehat{\mathcal{S}}$:

[6] $\lceil r\rceil$ is the least upper integer bound of r.

Definition 6 *The* aggregational ordering *of $\widehat{\mathcal{S}}$ is a subclass of partitions of $\widehat{\mathcal{S}}$, $\langle \Pi^{\alpha} \rangle_{\alpha \geq 0}$, such that $\Pi^0 = \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ and for every pair of indexes $\alpha < \alpha^{'}$, Π^{α} is a* refinement *of $\Pi^{\alpha^{'}}$, i.e., for every $\pi \in \Pi^{\alpha}$ there exists $\pi^{'} \in \Pi^{\alpha^{'}}$ such that $\pi \subseteq \pi^{'}$.*

Then we have the following result:

Theorem 1 *For every chain $\mathcal{C}$ in $\langle \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}, \preceq \rangle$ there exists a subsequence $\langle \Pi^{\alpha_0}, \Pi^{\alpha_1}, \ldots, \Pi^{\alpha_k}, \ldots \rangle_{k \geq 0}$ of $\langle \Pi^{\alpha} \rangle_{\alpha \geq 0}$, such that $\Pi^{\alpha_0} = \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)} = \Pi^0$ and for every pair $k < k^{'}$, $\alpha_k < \alpha_{k^{'}}$. Moreover, if $\mathcal{C} = \langle \mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^{|\mathcal{C}|} \rangle$, where $|\mathcal{C}|$ is the length of $\mathcal{C}$, is such that for every pair of indexes $k < k^{'}$, $\mathbf{f}^k \preceq \mathbf{f}^{k^{'}}$ then we have that every $\mathbf{f}^k$ is supported by a $\pi^k \in \Pi^{\alpha_k}$.*

Proof of Theorem 1 *First of all, let us note that each chain $\mathcal{C}$ in $\langle \langle \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}, \preceq \rangle$ has a* length*, defined as its cardinality and denoted as $|\mathcal{C}|$. Consider first the case in which $|\mathcal{C}| \leq \aleph_0$, i.e. that $\mathcal{C}$ is countable. If $\mathcal{C} = \langle \mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^{|\mathcal{C}|} \rangle$, $\mathbf{f}^0$ is the only prime aggregation function in $\mathcal{C}$. According to Lemma 1, for $\mathbf{f}^0$ there exists a $\bar{\mathcal{S}}^0 \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ such that $\mathbf{f}^0(prof(\mathcal{S})) = \bar{R}$ for every $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R} \rangle \in \bar{\mathcal{S}}^0$. We will prove by induction that for each $\mathbf{f}^k \in \mathcal{C}$ there exists an element π^k in a partition Π^{α^k} such that $\mathbf{f}^k(prof(\mathcal{S})) = \bar{R}$ for every $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R} \rangle \in \pi^k$. Moreover, that Π^{α^k} is a refinement of $\Pi^{\alpha^{k+1}}$ for each k:*

- *Consider the case of $\mathbf{f}^1$. According to Proposition 2 we have that there exists $\{\bar{\mathcal{S}}^j\}_{j \in J} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ such that $\mathbf{f}(prof_{\mathcal{S}}) = \bar{R}$ for every $\mathcal{S} = \langle S, prof_{\mathcal{S}}; \bar{R} \rangle \in \cup_{j \in J} \bar{\mathcal{S}}^j$. Suppose that $\bar{\mathcal{S}}^0 \neq \bar{\mathcal{S}}^j$ for every $j \in J$, but then this means that $\mathbf{f}^0(prof_{\mathcal{S}}) \neq \mathbf{f}^1(prof_{\mathcal{S}})$ for every $\mathcal{S} \in \bar{\mathcal{S}}^0$. Absurd, since we assumed that $\mathbf{f}^0 \preceq \mathbf{f}^1$. Then $\bar{\mathcal{S}}^0 \in \{\bar{\mathcal{S}}^j\}_{j \in J}$. That is, while $\mathbf{f}^0$ is supported by $\bar{\mathcal{S}}^0 \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$, $\mathbf{f}^1$ is supported by $\bar{\mathcal{S}}^0 \cup_{j \in J, j \neq 0} \bar{\mathcal{S}}^j$. If we call $\bar{\mathcal{S}}^0$, π^0 and $\widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$, Π^{α_0}, we see that if we denote $\bar{\mathcal{S}}^0 \cup_{j \in J, j \neq 0} \bar{\mathcal{S}}^j$ as π^1, it is clear that $\pi_0 \subseteq \pi_1$. On the other hand, $\pi_1 \cup_{j^{'} \notin J} \bar{\mathcal{S}}^{j^{'}} = \widehat{\mathcal{S}}$ while $\pi_1 \cap \bar{\mathcal{S}}^{j^{'}} = \emptyset$ for every $j^{'} \notin J$. That means that π_1 and $\{\bar{\mathcal{S}}^{j^{'}}\}_{j^{'} \notin J}$ constitute a partition of $\widehat{\mathcal{S}}$, which we call Π^{α_1}. Since for $\pi^0 \in \Pi^{\alpha^0}$, $\pi^0 \subseteq \pi^1$ and for each $\bar{\mathcal{S}}^j \in \Pi^{\alpha^0}$ $(j \in J)$, $\bar{\mathcal{S}}^j \subseteq \pi^1$ while $\bar{\mathcal{S}}^{j^{'}} \in \Pi^{\alpha^1}$ for $j^{'} \notin J$, Π^{α^1} is a coarsening of Π^{α^0}.*

- *Assume that the result is valid up to k. That is, $\mathbf{f}^k$ is supported by a $\pi^k \in \Pi^{\alpha^k}$. Without loss of generality we assume that if $\mathbf{f}^k$ is supported,*

according to Proposition 2, by a family $\{\bar{\mathcal{S}}^l\}_{l\in L} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *then* $\pi^k = \cup_{l\in L}\bar{\mathcal{S}}^l$ *and* $\Pi^{\alpha^k} = \{\pi^k, \{\bar{\mathcal{S}}^{l'}\}_{l'\notin L}\}$. *Now consider the case of* $\mathbf{f}^{k+1}$. *Again, by Proposition 2, we have that there exists* $\{\bar{\mathcal{S}}^j\}_{j\in J} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *such that* $\mathbf{f}^{k+1}(prof_{\mathcal{S}}) = \bar{R}$ *for every* $\mathcal{S} \in \cup_{j\in J}\bar{\mathcal{S}}^j$. *Since* $\mathbf{f}^k \preceq \mathbf{f}^{k+1}$ *we have that* $\mathbf{f}^k(prof_{\mathcal{S}}) \subseteq \mathbf{f}^{k+1}(prof_{\mathcal{S}})$ *for every* $\mathcal{S} \in \pi^k$. *That is,* $\pi^k \in \{\bar{\mathcal{S}}^j\}_{j\in J}$. *Let us call the latter* π^{k+1}. *We have that* π^{k+1} *and* $\{\bar{\mathcal{S}}^{j'}\}_{j'\notin J}$ *is a partition of* $\widehat{\mathcal{S}}$ *which we call* $\Pi^{\alpha^{k+1}}$. *It is clear that* $\pi^k \subseteq \pi^{k+1}$ *and* $\bar{\mathcal{S}}^j \subseteq \pi^{k+1}$ *for* $j \in J$, *while* $\bar{\mathcal{S}}^{j'} \in \Pi^{\alpha^{k+1}}$ *for* $j' \notin J$. *Therefore,* $\Pi^{\alpha^{k+1}}$ *is a coarsening of* Π^{α^k}.

Therefore, the claim is proved for every k.
Now consider the case in which $|\mathcal{C}| > \aleph_0$. *Then, the set of indexes* k *in* $\mathcal{C} = \langle \mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^{|\mathcal{C}|}\rangle$ *can be decomposed in two classes, those of the functions with either successor or limit indexes. That is, those indexes* k' *that have the form* $k' = k+1$ *(the* successor *indexes) and those which verify that* $k' = lim_{k<k'}k$, *the* limit *indexes. The proof for the countable length of* $\mathcal{C}$ *applies also for the successor indexes. The following is the proof for the limit indexes:*

- *Assume that the result is valid for each* $k < k'$ *for a limit index* k'. *That is, each* $\mathbf{f}^k$ *is supported by a* $\pi^k \in \Pi^{\alpha^k}$ *and, as* $\mathbf{f}^k$ *is supported by a family* $\{\bar{\mathcal{S}}^l\}_{l\in L} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *then* $\pi^k = \cup_{l\in L}\bar{\mathcal{S}}^l$ *and* $\Pi^{\alpha^k} = \{\pi^k, \{\bar{\mathcal{S}}^{l'}\}_{l'\notin L}\}$. *Moreover, for each pair* $k_1 < k_2 < k'$, $\Pi^{\alpha^{k_1}}$ *is a refinement of* $\Pi^{\alpha^{k_2}}$. *Let us now consider* $\mathbf{f}^{k'}$. *By Proposition 2 we have that there exists* $\{\bar{\mathcal{S}}^j\}_{j\in J} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ *such that* $\mathbf{f}^{k'}(prof_{\mathcal{S}}) = \bar{R}$ *for every* $\mathcal{S} \in \cup_{j\in J}\bar{\mathcal{S}}^j$. *For every* $\mathbf{f}^k \preceq \mathbf{f}^{k+1}$ *we have that* $\mathbf{f}^k(prof_{\mathcal{S}}) \subseteq \mathbf{f}^{k'}(prof_{\mathcal{S}})$ *for every* $\mathcal{S} \in \pi^k$. *That is, each* $\pi^k \in \{\bar{\mathcal{S}}^j\}_{j\in J}$. *Let us call the latter* $\pi^{k'}$. *We have that* $\pi^{k'}$ *and* $\{\bar{\mathcal{S}}^{j'}\}_{j'\notin J}$ *is a partition of* $\widehat{\mathcal{S}}$ *which we call* $\Pi^{\alpha^{k'}}$. *It is clear that for each* $k < k'$, $\pi^k \subseteq \pi^{k'}$ *and* $\bar{\mathcal{S}}^j \subseteq \pi^{k'}$ *for* $j \in J$, *while* $\bar{\mathcal{S}}^{j'} \in \Pi^{\alpha^{k'}}$ *for* $j' \notin J$. *Therefore,* $\Pi^{\alpha^{k'}}$ *is a coarsening of* $\{\Pi^{\alpha^k}\}_{k<k'}$.□

As discussed in Section 2, each equivalence class $\bar{\mathcal{S}} \in \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$ is defined by its decision sets, $\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}}) = \{\mathcal{D}(s,t)\}_{s,t\in S}$. Notice that each aggregation function $\mathbf{f}$, associated to a particular $\pi^{\mathbf{f}}$ in a partition $\Pi^{\mathbf{f}}$ of $\widehat{\mathcal{S}}$, is defined in terms of the decision sets corresponding to $\pi^{\mathbf{f}}$. More precisely, since

$\pi^{\mathbf{f}} \subseteq \widehat{\mathcal{S}}/\simeq_{\mathcal{D}(\mathcal{G}_n,\mathcal{G}_S)}$, there exist a set of indexes J such that $\pi^{\mathbf{f}} = \cup_{j\in J}\bar{\mathcal{S}}^j$ and therefore the decision sets of $\pi^{\mathbf{f}}$ are $\cup_{j\in J}\mathcal{F}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}}^j) = \cup_j\{\{\mathcal{D}(s,t)\}_{s,t\in S}\}^j$. A immediate consequence of Theorem 1 is that this set, which we call $\mathcal{DEC}(\mathbf{f})$ defines completely $\mathbf{f}$. In other words:

Proposition 3 *For every chain $\mathcal{C}$ in $\langle \mathbf{F}_{\mathcal{G}_n,\mathcal{G}_S}, \preceq\rangle$, $\mathcal{C} = \langle \mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^{|\mathcal{C}|}\rangle$, there exists a sequence $\mathcal{C}^{\mathcal{DEC}} = \langle \mathcal{DEC}(\mathbf{f}^0), \mathcal{DEC}(\mathbf{f}^1), \ldots, \mathcal{DEC}(\mathbf{f}^{|\mathcal{C}|})\rangle$, such that $\mathcal{DEC}(\mathbf{f}^k)$ are the decision sets defining $\mathbf{f}^k$. Moreover, for $k < k'$, $\mathcal{DEC}(\mathbf{f}^k) \subseteq \mathcal{DEC}(\mathbf{f}^{k'})$. That is, $\mathcal{C}^{\mathcal{DEC}}$ is a chain under set-theoretic inclusion.*

Proof of Proposition 3 *According to Theorem 1, $\mathcal{C}$ generates a chain of partitions, and particularly of equivalence classes, a π^k for each $\mathbf{f}^k \in \mathcal{C}$. Since each π^k is fully described by $\mathcal{DEC}(\mathbf{f}^k)$, we have that $\mathcal{C}$ generates also a sequence of the form $\mathcal{C}^{\mathcal{DEC}} = \{\mathcal{DEC}(\mathbf{f}^k)\}_{k=0}^{|\mathcal{C}|}$. To see that $\mathcal{C}^{\mathcal{DEC}}$ is a chain under $\subseteq$, just consider the fact that for $k < k'$, for every profile $prof_{\mathcal{S}}$ we have that $\mathbf{f}^k(prof_{\mathcal{S}}) \subseteq \mathbf{f}^{k'}(prof_{\mathcal{S}})$. Since, in particular $\mathbf{f}^k(prof_{\mathcal{S}})_{(s,t)} \subseteq \mathbf{f}^{k'}(prof_{\mathcal{S}})_{(s,t)}$ for each situation $\mathcal{S}$ and every pair $s,t \in S$, the class of agents that are decisive over s and t for the former function, $D^k(\{s,t\}) \in \mathcal{F}^k_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}}) \subseteq \mathcal{DEC}(\mathbf{f}^k)$ is also a class of decisive agents over s and t for the latter. That is, $D^k(\{s,t\}) \in \mathcal{F}^{k'}_{\mathcal{G}_n,\mathcal{G}_S}(\bar{\mathcal{S}}) \subseteq \mathcal{DEC}(\mathbf{f}^{k'})$. In other words: $\mathcal{DEC}(\mathbf{f}^k) \subseteq \mathcal{DEC}(\mathbf{f}^{k'})$.*□

4 Social Choice Criteria and Aggregation Functions

Since the structure of aggregation functions is paralleled by the structure of their corresponding decision sets we will concentrate on the later, as they summarize all the relevant information about the aggregation procedures. To be more precise, we will derive the axioms satisfied by any aggregation function up from the characteristics of its corresponding decision sets.

A first step towards that goal is to "normalize" the decision sets corresponding to each element in the partition:

Definition 7 *Given a aggregation function $\mathbf{f}$, if $\mathcal{S}$ and $\mathcal{S}'$ are in its support, $\mathcal{SUPP}(\mathbf{f})$, for every $s,t \in S$, there exist permutations $\rho \in \mathcal{G}_n$ and $\gamma \in \mathcal{G}_S$ such that their corresponding decision sets verify that $D^{\mathcal{S}'}(\{s,t\}) = \rho(D^{\mathcal{S}}(\{\gamma(s),\gamma(t)\}))$. Then*

$$\mathcal{D}(\{s,t\}) = \{\rho : \{1,\ldots,n\} \to \{1,\ldots,n\}, \gamma : S \to S \ :$$

$$D^{\mathcal{S}'}(\{s,t\}) \;=\; \rho(D^{\mathcal{S}}(\{\gamma(s),\gamma(t)\}));\; \mathcal{S},\mathcal{S}' \in \mathcal{SUPP}(\mathbf{f})\}$$

is the set of permutations that transform the decision set of $\mathbf{f}$ *for* s *and* t *over* $\mathcal{S}$ *into every other decision set for two alternatives over* $\mathcal{SUPP}(\mathbf{f})$. *For each* $D(\{s,t\}) \in \mathcal{DEC}(\mathbf{f})$ *we obtain the corresponding* $\mathcal{D}(\{s,t\})$. *Then the* quotient *of permutations under* $\mathcal{DEC}(\mathbf{f})$ *is, after choosing an arbitrary* $\mathcal{S} \in \mathcal{SUPP}(\mathbf{f})$,[7]

$$\mathbf{DEC}_{\mathcal{G}_n,\mathcal{G}_S}(\mathbf{f}) = \{\langle \mathcal{D}(\{s,t\}), D^{\mathcal{S}}(\{s,t\})\rangle : D(\{s,t\}) \in \mathcal{DEC}(\mathbf{f}), \mathcal{S} \in \mathcal{SUPP}(\mathbf{f})\}.$$

Even if $\mathbf{DEC}(\mathbf{f})$ is more complex than $\mathcal{DEC}(\mathbf{f})$, it allows us to get rid of the complications of using definitions *modulo* permutations. We have to see that $\mathbf{DEC}(\mathbf{f})$ *preserves* the properties of $\mathcal{DEC}(\mathbf{f})$, independently of the chosen $\mathcal{S}$. Since the only relevant property of any element of $\mathcal{DEC}(\mathbf{f})$ is to be a decision set, we have to check out that there always exist a pair of permutations in the first component of $\mathbf{DEC}(\mathbf{f})$ that, applied to its second component, verifies the condition of Definition 2. Formally:

Proposition 4 *If* $D^{\mathcal{S}'}(\{s,t\})$ *is a decision set for* $s,t \in S$ *under* $\mathbf{f}$ *on a situation* $\mathcal{S}' \in \mathcal{SUPP}(\mathbf{f})$, *then for every other situation* $\mathcal{S}'' \in \mathcal{SUPP}(\mathbf{f})$ *and every pair of permutations* ρ *and* γ *such that* $D^{\mathcal{S}'}(\{s,t\}) = \rho(D^{\mathcal{S}''}(\{\gamma(s),\gamma(t)\}))$ *there exist permutations* ρ',γ' *and* ρ'',γ'' *in* $\mathcal{D}(\{s,t\})$ *for* $\langle D(\{s,t\}), D^{\mathcal{S}}(\{s,t\})\rangle \in \mathbf{DEC}(\mathbf{f})$ *verifying that (over* $D^{\mathcal{S}'}(\{s,t\})$ *and* $D^{\mathcal{S}''}(\{s,t\})$*)* $\rho = \rho' \circ \rho''^{-1}$ *and* $\gamma = \gamma' \circ \gamma''^{-1}$ *where* ρ''^{-1} *and* γ''^{-1} *are the inverse permutations of* ρ'' *and* γ'', *respectively. Moreover, this property is independent of the choice of* $\mathcal{S}$ *in the definition of* $\mathbf{DEC}(\mathbf{f})$.

Proof of Proposition 4 *Trivial. By definition of* $\mathbf{DEC}(\mathbf{f})$ *there exist* ρ' *and* γ' *such that* $D^{\mathcal{S}'}(\{s,t\}) = \rho'(D^{\mathcal{S}}(\{\gamma'(s),\gamma'(t)\}))$ *and* ρ'' *and* γ'' *such that* $D^{\mathcal{S}''}(\{s,t\}) = \rho''(D^{\mathcal{S}}(\{\gamma''(s),\gamma''(t)\}))$. *The later implies that* $D^{\mathcal{S}}(\{s,t\}) = \rho''^{-1}(D^{\mathcal{S}''}(\{\gamma''^{-1}(s),\gamma''^{-1}(t)\}))$. *Then we have that*

$$D^{\mathcal{S}'}(\{s,t\}) = \rho'(D^{\mathcal{S}}(\{\gamma'(s),\gamma'(t)\})) =$$

$$\rho'(\rho''^{-1}(D^{\mathcal{S}''}(\{\gamma'(\gamma''^{-1}(s)),\gamma'(\gamma''^{-1}(t))\})))$$

Since $D^{\mathcal{S}'}(\{s,t\}) = \rho(D^{\mathcal{S}''}(\{\gamma(s),\gamma(t)\}))$ *it is clear that over* $D^{\mathcal{S}'}(\{s,t\})$ *and* $D^{\mathcal{S}''}(\{s,t\}))$, $\rho = \rho' \circ \rho''^{-1}$ *and* $\gamma = \gamma' \circ \gamma''^{-1}$.

[7] We will omit the subscript $\mathcal{G}_n,\mathcal{G}_S$, except when necessary.

If $\mathcal{S}$ is replaced, say, by $\mathcal{S}_0 \in \mathcal{SUPP}(\mathbf{f})$ it follows, from the argument above, that for ρ will exist $\rho_0^{'}$ and $\rho_0^{''}$ such that $\rho = \rho_0^{'} \circ \rho_0^{''-1}$ and $\gamma_0^{'}$, $\gamma_0^{''}$ such that $\gamma = \gamma_0^{'} \circ \gamma_0^{''-1}$. In other words $\rho^{'} \circ \rho^{''-1} = \rho_0^{'} \circ \rho_0^{''-1}$ and $\gamma^{'} \circ \gamma^{''-1} = \gamma_0^{'} \circ \gamma_0^{''-1}$ over $D^{\mathcal{S}^{'}}(\{s,t\})$ and $D^{\mathcal{S}^{''}}(\{s,t\})$). □

The structure of $\mathbf{DEC}(\mathbf{f})$ is straightforward: for every aggregation function $\mathbf{f}$, $\mathbf{DEC}(\mathbf{f}) \subseteq \mathcal{G}_n \times \mathcal{G}_S \times 2^n$. But this potentially enormous amount of possibilities can be substantially reduced if we note that the second component of $\mathbf{DEC}(\mathbf{f})$ bears most of the information that matters for the analysis of the structure of $\mathbf{f}$. That is, once we have chosen an arbitrary $\mathcal{S}$ in the support of $\mathbf{f}$, we know from the previous proposition that the decision sets over other elements in $\mathcal{SUPP}(\mathbf{f})$ can be obtained by applying a certain set of permutations on the decision sets corresponding to $\mathcal{S}$. Moreover, the particular choice of $\mathcal{S}$ does not change the characterization of the decisive sets for $\mathbf{f}$. Then, we can just focus our attention over $\{D^{\mathcal{S}}(\{s,t\}) : s,t \in S\}$. We define a structure $\mathcal{H}(\mathbf{f}) = \langle \{1,\ldots,n\}, \mathbf{DEC}(\mathbf{f})\rangle$, where the individuals are seen as "nodes", while each $D^{\mathcal{S}}(\{s,t\})$ is an "hyperedge" among the nodes, labeled both by the alternatives over which it is defined, $\{s,t\}$ and the class of permutations that preserve its decisiveness, $\mathcal{D}(\{s,t\})$. With this characterization, $\mathcal{H}(\mathbf{f})$ is a *labeled hypergraph* .[8]

Notice that Proposition 3 shows that $\mathcal{H}(\mathbf{f})$ is just a shorthand for the class of decision sets for $\mathbf{f}$. Therefore, it expresses in compact form how $\mathbf{f}$ behaves. That is, it indicates, for each pair of alternatives s,t, the individual preferences over s and t that prevail in the social ordering and which permutations leave them decision-invariant. It encodes, therefore, many of the relevant criteria that are satisfied by $\mathbf{f}$. An example may illustrate how this works:

Example 4 *Assume that we have $S = \{a,b,c\}$, and $n = 3$, while $\mathbf{f}$ is the dictatorial rule that imposes agent 1's will over the entire group. Any permutation that transforms the decision sets into other decision sets must therefore leave one agent invariant. That means that from $|\mathcal{G}_n \times \mathcal{G}_S \times 2^3| \leq 3! \times 3! \times 8 = 288$ possible permutations we can allow only those that "transform" one situation into a decision-equivalent one. A possible specification of the corresponding (unlabeled) hypergraph is, if $|\mathcal{G}_n| = 3! = |\mathcal{G}_S|$, $\mathcal{H}(\mathbf{f}) = \langle 3, \{\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}\rangle$. If, instead, $\mathcal{G}_n = \{Id_3\} = \mathcal{G}_S$, the only*

[8] In the case in which each hyperedge has a cardinality of 2 and dropping the labels we obtain the notion of *graph* as a particular case.

possible unlabeled hypergraph is $\mathcal{H}(\mathbf{f}) = \langle 3, \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}\rangle$. *In each case, just one individual is a member of each possible decision set.*

Two intuitions can be drawn from this example. The first is that, depending on $\mathcal{G}_n$ and $\mathcal{G}_S$ some information is lost when considering $\mathcal{H}(\mathbf{f})$, in particular anything that refers to the name of a particular alternative or individual. On the other hand, the general criteria satisfied by the aggregation function (like the existence of a dictator) can still be detected.

The criteria satisfied by $\mathcal{H}(\mathbf{f})$ can be conceived as related to the *axioms* in a formal system that defines the properties of the aggregation function. To make this more precise let us give some definitions:

Definition 8 *The* formal language for aggregation functions *over the set of alternatives* S *and* n *agents,* $\mathcal{L}_n^S$*, is a two-sorted first order language, and the class of its sentences is* $\mathbf{SENT}(\mathcal{L}_n^S)$.[9] *A theory of an aggregation function on this setting is* $\mathcal{T} \subseteq \mathbf{SENT}(\mathcal{L}_n^S)$ *such that* $\mathcal{T}$ *is* consistent*:* $\mathcal{T} \nvdash \bot$*. A set of* axioms *for* $\mathcal{T}$ *is* $\mathcal{A} \subseteq \mathcal{T}$ *such that* $\mathcal{A} \vdash \mathcal{T}$*.*

Example 5 *Consider the case of the aggregation function* $\mathbf{f}$ *over three alternatives and for three agents that imposes agent* 1*'s will over the entire group. We can denote this in FOL as:*

$$\phi^1 : \forall x, y \; R(1, x, y) \rightarrow \bar{R}(x, y)$$

where $\bar{R}(x, y)$ *is interpreted as the* more-than-or-equal *preferential ordering imposed by* $\mathbf{f}$*, on* x, y*, which are variables ranging over the set of alternatives* $\{a, b, c\}$*, while* $R(i, \cdot, \cdot)$ *is such that its first argument ranges over the set of agents, while the other two range over the alternatives. If we want to have a theory of* $\mathbf{f}$*, say* $\mathcal{T}^{\mathbf{f}}$*,* ϕ^1 *must be such that* $\phi^1 \in \mathcal{T}^{\mathbf{f}}$*. Moreover, since* ϕ^1 *characterizes the aggregation function,* $\phi^1 \in \mathcal{A}^{\mathbf{f}}$*, i.e. it is an axiom of the theory. On the other hand, it is clear that* $\mathbf{f}$ *verifies trivially the* Pareto condition*, which can be represented as:*

$$\phi^P : \forall x, y \; [R(1, x, y) \wedge R(2, x, y) \wedge R(3, x, y)] \rightarrow \bar{R}(x, y)$$

Then we have that $\phi^P \in \mathcal{T}^{\mathbf{f}}$ *and, moreover, that* $\phi^1 \vdash \phi^P$*. Instead, if we have that a formula like:*

$$\phi^{-1} : \forall x, y \; R(1, x, y) \rightarrow \neg \bar{R}(x, y)$$

[9] For a general treatment of first-order logic see Smullyan (1995). The specific features of $\mathcal{L}_n^S$ will be introduced in the next section.

stating that 1 *is an* antidictator*(i.e. what* 1 *chooses is never chosen by the society),* $\phi^{-1} \notin \mathcal{T}^{\mathbf{f}}$*, since otherwise* $\phi^1 \wedge \phi^{-1} \vdash \bot$*. A weaker version of* ϕ^{-1}*, namely*

$$\phi^{-1'} : \forall x, y \; R(1, x, y) \rightarrow \bar{R}(y, x)$$

is, instead, not *inconsistent with* $\mathcal{T}^{\mathbf{f}}$*.*

The *syntactic* characterization given in Definition 8 has a *model-theoretic* counterpart. That is, each sentence in $\mathcal{L}_n^S$ may be interpreted in such a way that a theory $\mathcal{T} \subseteq \mathbf{SENT}(\mathcal{L}_n^S)$ can be seen as a set of claims about the behavior of the constants and relations in the interpretation. If there exists an interpretation $\mathcal{M}$ that satisfies (makes true) every formula of $\mathcal{T}$ it is called a *model* of the theory. Moreover, if $\mathcal{T}$ is *consistent* it has at least one model.[10]

In our case, as the previous example shows, the interpretation of symbols in a theory about the behavior of an aggregation function is straightforward: variables and constants range over the sets of individuals and alternatives, $\{1, \ldots, n\}$ and $\mathcal{S}$ respectively, while any atomic expression $R(i, x, y)$ is interpreted as the preference of agent i over x, y, i.e. $R_i(x, y)$. Finally, if the interpretation is intended to be a model, $\bar{R}$ is to be interpreted as the social preference imposed by $\mathbf{f}$. Therefore, any theory $\mathcal{T}^{\mathbf{f}}$ of a aggregation function $\mathbf{f}$ has each situation $\mathcal{S} \in \mathcal{SUPP}(\mathbf{f})$ as one of its models. Moreover, if $\mathcal{S} = \langle S, R_1, \ldots, R_n; \bar{R} \rangle \notin \mathcal{SUPP}(\mathbf{f})$, we have by definition that $\mathbf{f}(\text{prof}(\mathcal{S})) \neq \bar{R}$. Then $\mathcal{S}$ is not a model of $\mathcal{T}^{\mathbf{f}}$.

Given any structure $\mathcal{M}$ we may look for a theory for which $\mathcal{M}$ is a model. If $\mathcal{M}$ has only a finite number of constants and relations defined over them, it is easy to find a finite set of sentences that can be interpreted to yield the true relations in the structure. Things become more interesting if we intend to find a theory corresponding to a given a class of structures $\bar{\mathcal{M}}$. In particular, we will try to find $\mathcal{T}^{\mathbf{f}}$ given $\mathcal{SUPP}(\mathbf{f})$.

As discussed above, the labeled hypergraph $\mathcal{H}(\mathbf{f})$ represents in summary form the information about $\mathcal{SUPP}(\mathbf{f})$. It indicates the decisive sets for every pair of alternatives and the permutations of agents that transform the decisive sets for a situation into the decisive sets in another situation in $\mathcal{SUPP}(\mathbf{f})$. This information is useful to derive $\mathcal{T}^{\mathbf{f}}$.

[10] This is why in Example 5, ϕ^1 and $\phi^{-1'}$ are not inconsistent: there exist a model where they are both true. In that model the social order is indifferent between any pair of alternatives.

5 Theories up from $\mathcal{H}(\mathbf{f})$s

As said, we expect to derive $\mathcal{T}^{\mathbf{f}}$ up from $\mathcal{H}(\mathbf{f})$ but first we have to show that this is feasible. Notice that $\mathcal{H}(\mathbf{f})$ yields, applying the permutations indicated in the labels of its hyperedges, a number of families of alternative decision sets over the same set of alternatives. This information can be used to reconstruct a class of social choice situations, namely those in $\mathcal{SUPP}(\mathbf{f})$. Each of those is a model for a collection of first order formulas $\mathcal{C}$. This formulas obtain just translating the corresponding relations described by $\mathcal{H}(\mathbf{f})$. Consider $\mathcal{L}_n^S$ as a two-sorted formal language with formulas build up from following elements:

- constants of the *agent* sort: $\bar{I} = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$,
- constants of the *alternative* sort: $\bar{A} = \{\bar{s}, \bar{t}, \ldots\}$, such that $|\bar{A}| = |S|$,
- a family of variables of the *agent* sort, $\hat{I}$, with elements $\hat{\imath}, \hat{\jmath}, \hat{k}$, etc.,
- a family of variables of the *alternative* sort, X, with elements denoted x, y, etc.,
- an *individual* propositional function, $R(\cdot, \cdot, \cdot)$ that ranges over $\bar{I} \times \bar{A} \times \bar{A}$,
- a *social* propositional function $\bar{R}(\cdot, \cdot)$ ranging over $\bar{A} \times \bar{A}$,
- a family of functions $\bar{\mathcal{G}}_S$ where each $\bar{\gamma} \in \bar{\mathcal{G}}_S$, $\bar{\gamma} : \bar{A} \to \bar{A}$ is a permutation of constants of the alternative sort,
- a family of functions $\bar{\mathcal{G}}_n$ where each $\bar{\rho} \in \bar{\mathcal{G}}_n$, $\bar{\rho} : \bar{I} \to \bar{I}$ is a permutation of constants of the agent sort,
- the usual connectives and quantifiers of first-order logic.

In this restricted language the information in $\mathcal{H}(\mathbf{f})$ can be immediately expressed as follows. Each hyperedge is, as said, $\mathbf{H} = \langle \mathcal{I}, \mathcal{P}_{\mathcal{S}}, \mathbf{G_n}, \mathbf{G_S} \rangle$, where $\mathcal{I}$ is the set of agents connected by the hyperedge, $\mathcal{P}_{\mathcal{S}} \in \mathcal{S} \times \mathcal{S}$ is the pair of states over which those agents are decisive in $\mathcal{DEC}(\mathbf{f})$. Finally $\mathbf{G_n} \subseteq \mathcal{G_n}$ and $\mathbf{G_S} \subseteq \mathcal{G_S}$ are the families of permutations that preserve the decisiveness of $\mathcal{I}$ over $\mathcal{P}_{\mathcal{S}}$. Then we can obtain a family of first order sentences in $\mathbf{SENT}(\mathcal{L}_n^S)$, which will be denoted $\Phi^{\mathbf{f}}$:

Definition 9 *For each hyperedge* $\mathbf{H}$ *of* $\mathcal{H}(\mathbf{f})$*, if* $\mathcal{P}_S = \langle a, b \rangle$ *then, for every* $i \in \mathcal{I}$*, every* $\rho \in \mathbf{G_n}$ *and every* $\gamma \in \mathbf{G_S}$*, the sentence*

$$\phi : R(\bar{\rho}(i), \bar{\gamma}(a), \bar{\gamma}(b)) \rightarrow \bar{R}(\bar{\gamma}(a), \bar{\gamma}(b))$$

is included in $\Phi^{\mathbf{f}}$*.*

Each $\mathcal{S} \in \mathcal{SUPP}(\mathbf{f})$ constitutes an interpretation of $\Phi^{\mathbf{f}}$ by construction. This family is the basis on which we will define $\mathcal{T}^{\mathbf{f}}$ as:

$$\mathcal{T}^{\mathbf{f}} \ = \ \{\mu \in \mathbf{SENT}(\mathcal{L}_n^S) : \Phi^{\mathbf{f}} \vdash \mu\}$$

Notice that the number of formulas in $\mathcal{T}^{\mathbf{f}}$ is infinite. Even if S is finite, if any $\mathcal{S} \in \mathcal{SUPP}(\mathbf{f})$ is an interpretation for a formula $\phi \in \mathcal{T}^{\mathbf{f}}$, it also interprets $\phi \vee \psi$, for *any* first order sentence ψ. On the other hand, since we assume that each $\mathcal{S} \in \mathcal{SUPP}(\mathbf{f})$ is a model for $\mathcal{T}^{\mathbf{f}}$ it follows that it is *consistent*.

On the other hand, consider two different situations $\mathcal{S}$ and $\mathcal{S}'$ drawn from $\mathcal{SUPP}(\mathbf{f})$. Since their corresponding decision sets are the same up to permutations as defined in $\mathbf{H}$ of $\mathcal{H}(\mathbf{f})$, there exists an isomorphism between $\mathcal{S}$ and $\mathcal{S}'$, through the corresponding permutations. Technically, $\mathcal{S}$ and $\mathcal{S}'$ are said *elementarily equivalent* models of $\mathcal{T}^{\mathbf{f}}$. A well known result in model theory is (Stigum 1990):

Completeness Theorem: *Let* $\mathcal{T}$ *be a consistent first order theory with language* $\mathcal{L}$*. Then,* $\mathcal{T}$ *is complete if and only if any two models of* $\mathcal{T}$ *are elementarily equivalent.*

Even if $\mathcal{T}^{\mathbf{f}}$ is well defined and has the desirable metamathematical properties of consistency and completeness, we are interested in finding an **axiomatization** for $\mathbf{f}$, which means a minimal number of formulas $\mathcal{A}^{\mathbf{f}} \subseteq \mathcal{T}^{\mathbf{f}}$ such that for every $\mu \in \mathcal{T}^{\mathbf{f}}$, $\mathcal{A}^{\mathbf{f}} \vdash \mu$. We say that $\mathcal{A}^{\mathbf{f}}$ is the set *axioms* of $\mathcal{T}^{\mathbf{f}}$.[11]

Two different cases may arise. One is when S and n are finite and the other when either one is infinite. Each involves a different treatment.

[11] We will restrict the meaning of $\mathcal{A}^{\mathbf{f}}$ to be the non-logical axioms of $\mathcal{T}^{\mathbf{f}}$. The logical axioms are those of first-order logic.

5.1 Finite number of agents and alternatives

In this case $\Phi^{\mathbf{f}}$ is also finite. It is easy to check out, by the properties of the hyperedge $\mathbf{H}$ from which it was drawn, whether a given sentence ϕ, $R(i,a,b) \rightarrow \bar{R}(a,b)$ is verified by every $\rho \in \mathcal{G}_n$ and every $\gamma \in \mathcal{G}_S$ or not. Then

- If ϕ is verified by every $\rho \in \mathcal{G}_n$ and every $\gamma \in \mathcal{G}_S$, then define $\phi^{\forall\forall}$: $\forall\bar{\rho}\ \forall\bar{\gamma}\ [R(\bar{\rho}(i),\bar{\gamma}(a),\bar{\gamma}(b)) \rightarrow \bar{R}(\bar{\gamma}(a),\bar{\gamma}(b))]$.[12]
- If ϕ is verified by every $\rho \in \mathcal{G}_n$ and by many but not all $\gamma \in \mathcal{G}_S$, define $\phi^{\forall\exists} : \forall\bar{\rho}\ \exists\bar{\gamma}\ [R(\bar{\rho}(i),\bar{\gamma}(a),\bar{\gamma}(b)) \rightarrow \bar{R}(\bar{\gamma}(a),\bar{\gamma}(b))]$.
- If ϕ is verified by every $\gamma \in \mathcal{G}_S$ and by not all $\rho \in \mathcal{G}_n$, define $\phi^{\exists\forall}$: $\forall\bar{\gamma}\ \exists\bar{\rho}\ [R(\bar{\rho}(i),\bar{\gamma}(a),\bar{\gamma}(b)) \rightarrow \bar{R}(\bar{\gamma}(a),\bar{\gamma}(b))]$.
- If ϕ is verified by some but not all $\rho \in \mathcal{G}_n$ and $\gamma \in \mathcal{G}_S$, define $\phi^{\exists\exists}$: $\exists\bar{\rho}\ \exists\bar{\gamma}\ [R(\bar{\rho}(i),\bar{\gamma}(a),\bar{\gamma}(b)) \rightarrow \bar{R}(\bar{\gamma}(a),\bar{\gamma}(b))]$.

This means that each hyperedge $\mathbf{H}$ can be represented by a sentence, denoted by $\phi_{\mathbf{H}}$ (either $\phi^{\forall\forall}_{\mathbf{H}}$, $\phi^{\forall\exists}_{\mathbf{H}}$, $\phi^{\exists\forall}_{\mathbf{H}}$ or $\phi^{\exists\exists}_{\mathbf{H}}$). Of course, some information is lost, but the decision structure is kept.

By a similar examination of the class $\{\phi_{\mathbf{H}}\}_{\mathbf{H}\in\mathcal{H}(\mathbf{f})}$, we obtain eight possible cases of the form

$$\phi^{\triangle} : \triangle_{\hat{I}}\hat{\imath}\ \triangle^1_X x\ \triangle^2_X y\ \phi_{\mathbf{H}}$$

where $\triangle_{\hat{I}}$, $\triangle^1_X$ and $\triangle^2_X$ are either $\forall$ or $\exists$.

It is immediate that:

Proposition 5 *The quantified sentences $\phi^{\triangle}$ and $\phi_{\mathbf{H}}$ entail the entire set of sentences $\Phi^{\mathbf{f}}$:*

$$\{\phi^{\triangle}\} \cup \{\phi_{\mathbf{H}}\}_{\mathbf{H}\in\mathcal{H}(\mathbf{f})} \ \vdash\ \Phi^{\mathbf{f}}.$$

This result indicates that $\mathcal{A}^{\mathbf{f}} = \{\phi^{\triangle}\} \cup \{\phi_{\mathbf{H}}\}_{\mathbf{H}\in\mathcal{H}(\mathbf{f})}$ provides a finite *axiomatization* of $\mathcal{T}^{\mathbf{f}}$. This family of axioms $\mathcal{A}$ conveys the minimal essential information about social situations, while the full information is provided by $\mathcal{T}^{\mathbf{f}}$.

The set of axioms $\mathcal{A}^{\mathbf{f}}$ can be seen in the light of the following classification (Fishburn 1987):

[12]Notice that this sentence is still first order, since the quantifiers do not range over *propositional* (i.e. truth-valued) functions.

- *Structural conditions.* They prescribe conditions on the domain of aggregation (individuals, alternatives and profiles). The structural conditions just indicate the elements on which the aggregation procedure will be applied. Easy examples are the requirement that all individual preferential orderings have to be weak orders or that the number of individuals has to be larger than 2.

- *Existential conditions.* They prescribe the existence of certain results under an aggregation procedure. So, for example, an existential condition is that there should not exist an individual such that for every profile her preferences coincide with those that result from the aggregation procedure.[13]

- *Universal conditions.* They specify general aspects of the aggregation procedure. The universal conditions can be further classified as *intraprofile* conditions, which involve no comparisons among profiles or *interprofile* ones. An example of intraprofile conditions are the Pareto condition, which indicates that if every individual in a profile prefers more one alternative over another one, this feature must result also in the social ordering. An interprofile condition is, instead, the *independence of irrelevant alternatives*, which prescribes that if in two profiles the preferences of the agents with respect to any pair of alternatives are the same, then the order over these alternatives in the respective social orderings must coincide.

It follows that:

- The joint characterization of S, n and the corresponding $\mathcal{G}_S$ and $\mathcal{G}_n$ yields the *structural conditions* on the aggregation function $\mathbf{f}$.

- The sentences $\phi_{\mathbf{H}}$ yield the interprofile conditions.

- The sentence $\phi^{\triangle}$ summarizes both the interprofile and intraprofile conditions.

This indicates that to disentangle the intraprofile from the interprofile conditions, more information is needed, which can be found in $\Phi^{\mathbf{f}}$. The family of axioms $\mathcal{A}^{\mathbf{f}}$, provides only the most general properties of $\mathbf{f}$ while others can be derived from them. A simple example shows this:

[13]Notice that this *non-dictatorship* condition can be rewritten as to become the claim of the existence of a pair of alternatives which for every individual and every profile verify that the social outcome differs from the individual orderings.

Example 6 (Examples 4 and 5 revisited) *Consider either one of the unlabeled hypergraphs of Example 4, say* $\mathcal{H}(\mathbf{f}) = \langle 3, \{\{1\}, \{1,2\}, \{1,3\}\}\rangle$. *It constitutes an* ultrafilter*, i.e. every subset* $A \subseteq \{1,2,3\}$ *either* $A \in \mathcal{H}(\mathbf{f})$ *or* $A^C \in \mathcal{H}(\mathbf{f})$*, where* $A^C = \{1,2,3\} \setminus A$*. As* $\{1,2,3\}$ *is finite,* $\mathcal{H}(\mathbf{f})$ *is a* principal ultrafilter*. That is,* $\bigcap \mathcal{H}(\mathbf{f})$ *is a singleton. The element in this singleton denotes a dictator.*[14] *Then, the only* $\phi^{\triangle}$ *axiom that can be derived is, as in Example 5,* $\phi : \exists \hat{\imath}\ \forall \mathbf{x}, \mathbf{y}\ \mathbf{R}(\hat{\imath}, \mathbf{x}, \mathbf{y}) \rightarrow \bar{\mathbf{R}}(\mathbf{x}, \mathbf{y})$*. In fact,* $\mathcal{A}^{\mathbf{f}} = \{\phi\}$*. Notice that the variable* $\hat{\imath}$ *might be interpreted as being either one of the possible agents,* $1, 2$ *or* 3*. That is, in this first order representation the information about the* actual *dictator is lost.*

On the other hand, to get a theory that just implies the $\Phi^{\mathbf{f}}$, instead of one as general as $\mathcal{A}^{\mathbf{f}}$, we can just take $\Phi^{\mathbf{f}}$ as our axioms. Since they are finite, the axiomatization is of course also finite. But this procedure has a huge cost associated to it: all the advantages of extracting intensional from extensional information is lost. It is like if we were running a statistical procedure and to obtain an exact characterization of the sample we just enumerate the elements in it!

5.2 Infinite number of agents or alternatives

This case is far more complex. Notice that the language needed is no longer finitary, because of the number of constants required to represent the infinite elements in the domain. While this is a handicap, some results will allow us to recover a characterization of the set of axioms $\mathcal{A}^{\mathbf{f}}$ similar to the presented in the previous section. Notice that those axioms are expressed with just a few variables. Therefore, we can keep using a finitary fragment of $\mathcal{L}_n^S$ with the proviso that it will be used only to write down $\mathcal{A}^{\mathbf{f}}$.

Recall that $\mathcal{H}(\mathbf{f})$ is a hypergraph. We seek a family of sentences $\mathcal{A}^{\mathbf{f}}$ that can be **satisfied** by $\mathcal{H}(\mathbf{f})$:

Definition 10 $\mathcal{A}^{\mathbf{f}}$ *is satisfied if and only if there exists* $\sigma = \langle \mathcal{I}, \mathcal{P}_{\mathcal{S}}, \mathbf{G_n}, \mathbf{G_S}\rangle$*, where* $\mathcal{I} \subseteq \mathcal{I}$ *is a set of agents,* $\mathcal{P}_{\mathcal{S}} \in \mathcal{S} \times \mathcal{S}$ *is a pair of states, while* $\mathbf{G_n} \subseteq \mathcal{G_n}$ *and* $\mathbf{G_S} \subseteq \mathcal{G_S}$ *are permutations over* n *and* S*, that verifies (Kolany 1993):*

- **consistency***:* $\mathbf{H} \subset \sigma$ *for no* $\mathbf{H} \in \mathcal{H}(\mathbf{f})$*,*
- **transversality***:* $\sigma \cap \mathcal{INT}(\alpha) \neq \emptyset$*, for every sentence* $\alpha \in \mathcal{A}^{\mathbf{f}}$*, where* $\mathcal{INT}(\alpha)$ *is a hyperedge that yields a true interpretation of* α*.*

[14] See Kirman & Sondermann (1972), Brown (1974) or Hansson (1976).

The notion of satisfaction over hypergraphs is in fact equivalent to the existence of a **system of distinct representatives**, i.e. to the existence of a *transversal* σ that has an element from each hyperedge. The existence of such a σ is ensured by the following result:

Theorem(Hurwicz & Reiter 2001): $\mathcal{H}(\mathbf{f})$ has a transversal σ if and only if $\mathcal{H}(\mathbf{f})$ is generated by a *self-belonging* correspondence $\Upsilon : \bigcup \mathcal{H}(\mathbf{f}) \rightarrow \bigcup \mathcal{H}(\mathbf{f})$.

Two are the conditions on $\Upsilon : \bigcup \mathcal{H}(\mathbf{f}) \rightarrow \bigcup \mathcal{H}(\mathbf{f})$ that may ensure the existence of σ:

- *generation*: for every $\mathbf{H} \in \mathcal{H}(\mathbf{f})$, there exists $\theta \in \bigcup \mathcal{H}(\mathbf{f})$ verifying that $\mathbf{H} = \boldsymbol{\Upsilon}(\theta)$,
- *self-belonging*: for every $\theta \in \bigcup \mathcal{H}(\mathbf{f})$, $\theta \in \Upsilon(\theta)$.

In fact, it follows that:

Proposition 6 *There exists Υ that ensures the existence of a transversal σ.*

Proof of Proposition 6 *Trivial. By construction, each $\theta \in \bigcup \mathcal{H}(\mathbf{f})$ is $\theta = \langle i, \mathcal{P}_{\mathcal{S}}, \mathbf{G_n}, \mathbf{G_S} \rangle$, i.e. a single individual and a pair of states over which she is decisive(plus the permutations that preserve her decisiveness). Then $\theta \in \mathbf{H}$ for a $\mathbf{H} \in \mathcal{H}(\mathbf{f})$. Define then $\Upsilon(\theta)$ as the corresponding $\mathbf{H}$.* □

Finally:

Lemma 2 *There exists a set of sentences $\mathcal{A}^{\mathbf{f}}$ in $\mathcal{L}_n^S$ that is satisfied by the transversal σ. Moreover, each $\alpha \in \mathcal{A}^{\mathbf{f}}$ can be either $\phi_{\mathbf{H}}^{\forall\forall}$, $\phi_{\mathbf{H}}^{\forall\exists}$, $\phi_{\mathbf{H}}^{\exists\forall}$, $\phi_{\mathbf{H}}^{\exists\exists}$ or $\phi^{\triangle}$.*

Proof of Lemma 2 *The first claim follows immediately from Definition 10 and Proposition 6. The characterization of the sentences in $\mathcal{A}^{\mathbf{f}}$ follows from the fact that sentences of the type $\phi_{\mathbf{H}}^{\forall\forall}$, $\phi_{\mathbf{H}}^{\forall\exists}$, $\phi_{\mathbf{H}}^{\exists\forall}$, $\phi_{\mathbf{H}}^{\exists\exists}$ or $\phi^{\triangle}$ can verified in $\mathcal{H}^{\mathbf{f}}$ independently of S and n being finite or not.* □

Therefore, a similar set of axioms can be obtained for a potential theory $\mathcal{T}^{\mathbf{f}}$ as in the finite case. However, as in the former instance, the axioms of $\mathcal{A}^{\mathbf{f}}$ while conveying the general properties of f do not pinpoint their precise

details. Recalling that $\Phi^{\mathbf{f}}$ is now an infinite enumeration of sentences without quantifiers, we could apply the following theorem (Shoenfield 1967):

Theorem(Łos-Tarski) *A theory $\mathcal{T}$ is equivalent to a class of quantifier-free sentences Φ if and only if every substructure of a model of $\mathcal{T}$ is a model of $\mathcal{T}$.*

This means, in our case that a $\mathcal{T}^{\mathbf{f}}$ which yields precisely those sentences in $\Phi^{\mathbf{f}}$ and only those, must be such that its models have a "Russian doll" structure. That is, for each model its substructures are also models. As in the finite case, if we take the axioms of $\mathcal{T}^{\mathbf{f}}$ to be $\Phi^{\mathbf{f}}$, we have a trivial instance of Łos-Tarski's theorem. But in this case, added to the handicap of identifying the intensional characterization with the extensional enumeration, we have that the axiom system will no longer be finite.[15]

6 Conclusions

In this paper we presented an exercise in inductive reasoning, namely, on how to infer the criteria verified by social choice functions, up from the observable behavior of their decision sets. We have shown that social choice functions keep a close relation with partitions of social choice situations. Moreover, each social choice function yields a corresponding labeled hypergraph. The information of the later can be used to find out a minimal number of first order formulas verified by each instance of decision sets in the hypergraph. Those formulas constitute the axioms of a possible theory of the social choice function.

As already said, alternative characterizations of the equivalence among social choice situations may lead to related results. But the most interesting extension of this work, we feel, is to provide a framework for the detection of properties that could be represented in the derived axiomatic systems. The downside of this is that some other properties may get lost through the procedure of inferring theories up from hypergraphs.

An additional feature of the framework presented in this paper is that it yields more than a full-blown theory, a few very general axioms that may be useful to describe the properties of social choice functions. As usual in inductive inference, the information extracted by the procedure fleshes out the specific details of the characterization of the functions, leaving only their

[15] In fact, nothing in the theorem ensures that the axiomatization will be finite.

bare bones. By doing this, it erases the differences among similar functions, but at the same time it yields what we entertained originally to find: a few properties that are satisfied by them.

To fulfill the goal of providing a converse to the usual procedure in Social Choice Theory, we need to complete the axiomatic system with specific axioms. That is, ones that capture the details lost in our more general approach. It is obvious that in the finite case this is achieved by just enumerating all the formulas that translate the corresponding hypergraph. In the infinite case this is no longer possible, since the number of those formulas is no longer finite. But even in the finite case, to just keep the sentences that describe the hypergraph runs contrary to our idea of extracting intensional from extensional information. Of course, a specific knowledge of the function may help to find the required axioms. But this would no longer constitute a general procedure and therefore falls short of our goals.

This research points out, also, to a more general line of inquiry, namely the extraction of intensional information up from an extensional description in partitional form. While the method described is not independent from the underlying problem, many of its features are modular and could be applied to other problems in Economic Theory.

References

[Arrow 1963] Arrow, K.: *Social Choice and Individual Values*, Wiley, New York.

[Brown 1974] Brown, D.: An Approximate Solution to Arrow's Problem, *Journal of Economic Theory* **9**: 375–383.

[Campbell & Kelly 1997] Campbell, D. and Kelly, J.: Preference Aggregation, *Mathematica Japonica*, **45**: 573–593.

[Fishburn 1987] Fishburn, P.:*Interprofile Conditions and Impossibility*, Harwood Academic Publishers, Chur (Switzerland).

[Hansson 1976] Hansson, B.: The Existence of Group Preferences, *Public Choice* **28**: 89–98.

[Hurwicz & Reiter 2001] Hurwicz, L. and Reiter, S.: Transversals, Systems of Distinct Representatives, Mechanism Design and Matching, *Review of Economic Design* **6**: 289-304.

[Kirman & Sondermann 1972] Kirman, A. and Sondermann, D.: Arrow's Theorem, Many Agents and Invisible Dictators, *Journal of Economic Theory* **5**: 267–277.

[Kolany 1993] Kolany, A.: On the Logic of Hypergraphs, in **Computational Logic and Proof Theory**, Springer-Verlag, Berlin.

[Moulin 1994] Moulin, H.: Social Choice, in *Handbook of Game Theory* **II**, R. Aumann and S. Hart (eds.), North-Holland, Amsterdam.

[Lauwers & Van Liedekerke 1995] Lauwers, L. and Van Liedekerke, L.: Ultraproducts and Aggregation, *Journal of Mathematical Economics* **24**: 217–237.

[Rosen 2002] Rosen, E.: Some Aspects of Model Theory and Finite Structures, *Bulletin of Symbolic Logic* **8**: 380–403.

[Scott & Suppes 1958] Scott, D. and Suppes, P.: Foundational Aspects of Theories of Measurement, *Journal of Symbolic Logic* **23**: 113–128.

[Sen 1970] Sen, A.: *Collective Choice and Social Welfare*, Holden Day, San Francisco, CA.

[Shoenfield 1967] Shoenfield, J.: *Mathematical Logic*, Addison-Wesley, Reading, MA.

[Smullyan 1995] Smullyan, R.: *First-Order Logic*, Dover, New York.

[Stigum 1990] Stigum, B.: *Toward a Formal Science of Economics*, MIT Press, Cambridge, MA.